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THE MATHEMATICAL THEORY OF A NEW RELATIVITY

BY SIR SHAH SULAIMAN,

CHIEF JUSTICE, HIGH COURT, ALLAHABAD

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CHAPTER I

A New Relativity

SECTION I

INTRODUCTORY

Newton assumed that gravitation had an instantaneous effect, howsoever distant the object might be. This implied that its velocity was infinite. He further assumed that the same law of gravitation applied between two bodies, whether they were at rest or in relative motion. Later observations showed that his law was inaccurate for moving bodies.

Einstein has given a slightly more accurate law, but at the complete sacrifice of the principles of Newton. Relativity denies the absoluteness of space, time and motion, but can hardly deny the absoluteness of angular motion or sudden change of motion.

> "In the modern theory the question between Copernicus and his predecessors is merely one of convenience; all motion is

> relative, and there is no difference between the two statements: the earth rotates once a day and the heavens revolve round the earth once a day. The two mean exactly the same thing." (Bertrand Russell)[1]

But when a boy spins a top, does he give an absolute rotational motion to the top or does he set the entire universe revolving round the top in the opposite direction? Or again, when a motorist suddenly puts on the brakes, does he stop his car or does he push the whole universe on a backward path?

Relativity makes the velocity of light absolute, and although it is a known finite velocity (300,000 km. per sec.), the properties of infinity are attributed to it; and no velocity, howsoever great, when added to it or subtracted from it, can ever make any difference.

> "A Beta particle shot off from Radium can move at more than 200,000 km. per sec., but the speed of light relative to an observer travelling with it is still 300,000 km. per sec." (Eddington)[2]

It makes time and mass depend on velocity, and yet it is immaterial whether the body is approaching to or receding from the observer.

> "If A and B were twin brothers, then B must be younger when he returns from his voyage than A. This is truly a strange deduction, which can, however, be eliminated by no artificial quibbling. We must put up with this." (Max Born)[3]

It makes length contract in the direction of motion, creating the paradox of a rotating wheel, where the rim contracts, but the spokes do not in the same exact proportion.

Relativity does not only regard time as a fourth co-ordinate, but makes it a fourth dimension of space, and thereby gives to space a curvature and other properties, though still regarding it as a vacuum and not a medium like ether. It makes space finite, and yet makes its finite limit incapable of being reached except in infinite time, by making time itself slow down with distance, and ultimately become stationary. It extends the Pythagorean theorem to four dimensions and assumes that the square of the interval between two events is the difference between the square of the spacial dimension and the square of the distance travelled by light.

> "If you had been a ray of light, travelling round the solar system, starting from London at 10 a. m., reflected from Jupiter to

> Saturn, and so on, until you were reflected back to Edinburgh and arrived there at 6-30 p. m., you would judge that the journey had taken you exactly no time." (Russell)[4]

It assumes that all laws of nature must have an invariable form in vacuum only, and yet applies the principle to gravitational space. It has also introduced a cosmical force of repulsion between two bodies, which not only acts at a distance without any medium, but has its intensity incomprehensibly increasing as the distance between them increases, with the result that the whole universe must be exploding at a terrific rate. But the Galactic system, as Nature's favourite, is exempt from the operation of that law; for distant stars are not seen to be scattering away from one another at velocities proportional to the distances between them. Unfortunately for Relativity, at least five of the nebulae are approaching with rather high velocities, which are exceptionally well determined, and whose motions are not wholly explained by the rotation of the Galactic system.

> "In the full formula there are no terms which under any reasonable conditions encourage motion towards the origin. It is, therefore, difficult to account for these motions even as exceptional phenomena." (Eddington)[5]

> "The five approaching velocities are at least *partly* attributable to the use of an inappropriate standard of reference . . . . the approaching velocities are *reduced* or disappear." (Eddington)[6]

But even if they disappear they would not give recessional velocities proportional to distances.

As relativity compels every body to keep its own separate and independent time, the problem of the interaction of more than two bodies moving simultaneously becomes insoluble.

> "The existence of an electron contradicts the electromagnetic laws with which we have to work at present, so that from the present standpoint an electron at rest in no external field of force is a *miracle*. . . . . . An electron in an external field of force having the (derived) acceleration is precisely the *same miracle*." (Eddington)[7]

These apparently unconvincing assumptions remained unaccepted for many years, until Einstein's equations were believed to have been verified in three remarkable instances, *viz*., (*a*) the deflection of light from a star when passing close to the Sun, (*b*) the displacement of the

Fraunhofer lines and (*c*) the advance of the perihelion of Mercury. But for such verifications it is doubtful whether relativity could have held its ground so long. In the absence of any better substitute, there has been no option but to accept it. But it is now established that the supposed verifications are not exact.

(*a*) According to Newton's theory, the deflection of light should be $0''\cdot87$; according to Einstein, it should be $1''\cdot75$; but Freundlich and Klüber in 1931 found it to be $2''\cdot24 \pm 0''\cdot10$ if not more[8].

(*b*) The displacement of the blue light according to Einstein should be ·0084; but it is only ·0036, as observed by St. John[9].

The recent observation of Vyssotshiy[10] that the density of the Companion of Sirius is less, and therefore its radius larger than what was supposed, creates a similar discrepancy between Einstein's theoretical value and the observed value.

(*c*) Einstein's value for the advance of the perihelion is $42''\cdot9$; and this theoretical value had been found even before Relativity by Gerber[11] in 1902; but his method was different and his equation did not yield the value for the deflection of light. The calculated mean value based on observation comes to $40''\cdot00$[12].

If it can be shown that the ordinary principles of dynamics, when applied to moving bodies, themselves yield modified forms of equations, which as a first approximation reduce themselves to Newton's forms, and as a second approximation to Einstein's forms, the Newtonian mechanics would be restored to the eminent position it occupied before its dethronement by Relativity, and there would no longer be any absolute necessity to accept the extraordinary hypotheses on which Relativity is founded. If the theoretical values derived from the equations tally more exactly with the observed values, and the deflection of light were 2·6 times that under Newton's law, the displacement of the Fraunhofer lines were less than ½ of the value under Relativity, the theoretical value for the advance of the perihelion nearer to the observed value, and velocities of both recession and approach were permissible for the nebulae, the assumptions in Relativity would be proved to be wrong. It is submitted herein that such modified forms of equations can be obtained.

I must express my gratitude to Dr. D. S. Kothari, M. Sc., Ph.D., Reader in Physics at the Delhi University, for his kindness in going through the manuscript and making valuable suggestions, and also to Mr. A. N. Chatterji, M.Sc., who has kindly helped me in checking the mathematical processes and making calculations.

SECTION II

GRAVITATION BETWEEN TWO STATIONARY BODIES

1. In the Unified Theory of Physical Phenomena a new theory of gravitation based on internal action and not due to any extraneous force acting at a distance was put forward. The assumption was that light consists of material corpuscles called "radions" which are emitted from surfaces of bodies, and that there are still finer particles called "gravit ons" which emanate from the entire mass, but are at present beyond the range of our perception. The rate of emanation of gravitons is dependent on the material density existing in the neighbourhood. It was shown that if the presence of matter retards emanations then the diminution in the losses of momenta from the side of a body facing another body as compared with the unaffected losses on the other side would result in a net difference of momentum in the direction of the influencing body, and would cause the influenced body to move towards it. This retardation of emanation from the side opposed to the moving gravitons can be interpreted as a stimulation of emanation in the direction of the moving gravitons as Einstein has assumed in the case of light. It was further shown that the apparent force between the two bodies must be proportional to their masses, and that the intensity of their mutual influence must vary inversely as the square of the distance between them, giving the Newtonian law

$$F = G.\frac{M.\,m}{R^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2{\cdot}1)$$

which holds between two *stationary* bodies only. This will be examined again mathematically in Chapter IV.

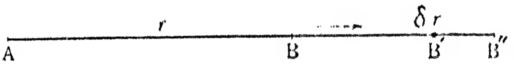
2. For the purposes of the matters dealt with in Chapter I, it is not even necessary to assume the existence of material gravitons. The results can be deduced equally well on the supposition that gravitation is the result of a wave propagation travelling with a finite velocity. For only portions of Chapters II and III the material aspects of radions and gravitons become necessary. It is only in Chapter IV when showing how both velocities of recession and approach become possible for nebulæ that the theory of emission of gravitons becomes important.

SECTION III

GRAVITATION ON A BODY MOVING RELATIVELY TO ANOTHER IN THE LINE OF CENTRES

1. It is submitted that Newton's law of gravitation which holds between two stationary bodies cannot apply to moving bodies. The apparent force of attraction between two bodies at rest is not the same when they are moving. Newton assumed that the effect of gravitation was instantaneous; but Laplace and several others after him assumed that it had a finite velocity. It is assumed herein that the velocity of gravitons is D which is nearly equal to C.

2. Let A and B be two small particles, A being supposed to be fixed and B moving relatively to it, and let the distance between them be r. If B also were stationary a graviton from A will reach B in time $T_1 = \frac{r}{D}$, when travelling



with velocity D. But if B moves towards B'', then by the time the graviton reaches the old position of B, the latter would have moved on to some intermediary position B'. Suppose the graviton overtakes B at the position B'' at a distance δr from B ; δr is positive when measured away from A and negative when measured towards it. Then the time taken by the graviton to reach B'' would be $T_2 = \frac{r+\delta r}{D}$. Then as successive gravitons reach B'' the frequency of gravitons would be changed in the ratio

$$\frac{v_2}{v_1} = \frac{T_1}{T_2} = \frac{r}{r+\delta r} = \frac{1}{1+\frac{\delta r}{r}} \quad \cdots \quad \cdots \quad \cdots \quad (3\cdot 1)$$

This is the result of the motion of B.

SECTION IV

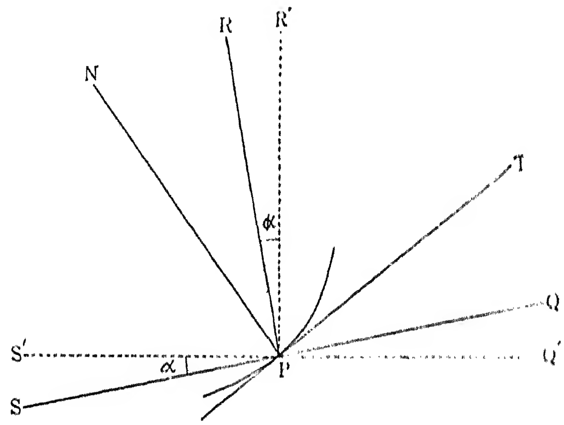
1. Applying the principle of aberration of light to gravitation, a necessary result of the finiteness of the velocity is to shift forward the line of attraction towards the direction of motion. If α be the angle of aberration ; θ the angle which the

ratio $\left(1-\frac{v}{D}\right)^3$, while the effect of its transverse velocity $r\frac{d\theta}{dt}$ is to shift the direction of attraction forward by an angle α, where $\sin\alpha = \frac{1}{D}\frac{rd\theta}{dt}$ (4·4)

It is also obvious that although the apparent direction is altered the number of gravitons reaching it per unit time must remain the same.

SECTION V
PLANETARY ORBIT

1. Let P be the position of a planet in its orbit and (r, θ) its polar coordinates with respect to the sun S. Let PS′ be the shifted direction of the attraction of the sun, making an angle α with PS. PR is the normal to SP and PR′ is normal to S′P.



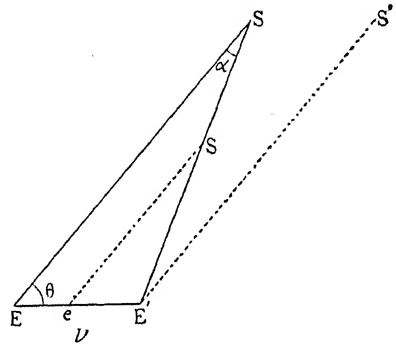
The accelerations due to the orbital motion along the radius vector and its normal are $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$ respectively.

But the whole force of attraction is directed along PS′ and there is no force along the normal PR′. The effect of the Doppler principle as given in (4·3) is to change the effective force of gravitation to $-\frac{\mu}{r^2}\left(1-\frac{v}{D}\right)^3$ where v is the velocity of the planet along S′P.

$$\text{But } v = \frac{dr}{dt}\cos\alpha - \frac{rd\theta}{dt}\sin\alpha.$$

$$= \frac{dr}{dt} - \frac{1}{D}\left(r\frac{d\theta}{dt}\right)^2$$

direction of the earth's motion makes with the real direction of a star and v the velocity of the earth then $\frac{v}{D}=\frac{\sin\alpha}{\sin\theta}$. When the earth is moving at right angles to the line, then $\theta=\frac{\pi}{2}$ and $\frac{v}{D}=\sin\alpha$, as also $\tan\alpha$ when α is small...(4·1)



2. On the other hand, the change in the force of attraction due to the displacement of B to B″ when it is overtaken by the gravitons is given by the ratio

$$\frac{\mu}{(r+\delta r)^2} : \frac{\mu}{v^2} = \frac{1}{\left(1+\frac{\delta r}{r}\right)^2}$$

It follows that the total change in the effect of gravitation due to both motion and displacement is given by the ratio

$$\frac{1}{\left(1+\frac{\delta r}{r}\right)^3} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (4{\cdot}2)$$

3. But as the times taken by the graviton to travel from A to B″ and by the particle to move from B to B″ are the same, $\frac{r+\delta r}{D} = \frac{\delta r}{v}$ where v is the velocity of the particle B. From this we get

$$\left(\frac{1}{1+\frac{\delta r}{r}}\right) = \left(1-\frac{v}{D}\right)$$

The increased ratio of the change of gravitational force is therefore $\left(1-\frac{v}{D}\right)^3$ which can be written in the more convenient form

$$\left(1-\frac{1}{D}\frac{dr}{dt}\right)^3 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (4{\cdot}3)$$

where $\frac{dr}{dt}$ will be positive when B is moving away from A, and negative when moving towards it.

4. It follows that when a planet is moving in its orbit round the sun, the effect of its radial velocity v is to change the effect of gravitation in the

since $\cos\alpha = 1$ nearly, and $\sin\alpha = \frac{v}{D} = \frac{1}{D}\, r\frac{d\theta}{dt}$.

Resolving the accelerations along S′P and PR′ we get

$$\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\cos\alpha - \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\sin\alpha$$

$$= -\frac{\mu}{r^2}\left[1 - \frac{1}{D}\left\{\frac{dr}{dt} - \frac{1}{D}\left(r\frac{d\theta}{dt}\right)^2\right\}\right]^3 \quad \ldots \quad \ldots \quad (5{\cdot}2)$$

and $$\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\sin\alpha + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\cos\alpha = 0 \quad \ldots \quad (5{\cdot}3)$$

Multiplying (5·2) by $-\sin\alpha$ and adding the result to (5·3) multiplied by $\cos\alpha$, we get

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\mu}{r^2}\left[1 - \frac{1}{D}\left\{\frac{dr}{dt} - \frac{1}{D}\left(r\frac{d\theta}{dt}\right)^2\right\}\right]^3 \sin\alpha. \quad \ldots \quad (5{\cdot}4)$$

Substituting $\frac{1}{D}\left(r\frac{d\theta}{dt}\right)$ for $\sin\alpha$ in (5·4) and neglecting terms of the second order on the right hand side after expansion we get,

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\mu}{D}\frac{d\theta}{dt}\left[1 - \frac{3}{D}\left\{\frac{dr}{dt} - \frac{1}{D}\left(r\frac{d\theta}{dt}\right)^2\right\}\ldots\right]$$

$$= \frac{\mu}{D}\frac{d\theta}{dt} \text{ nearly, the other terms being small} \quad \ldots \quad (5{\cdot}41)$$

Hence $$\frac{r^2 d\theta}{dt} = h\left(1 + \frac{\mu}{hD}\theta\right) \text{ where } h \text{ is a constant.}$$

$$= h\,(1 + k\,\theta) \text{ where } k = \frac{\mu}{hD} \quad \ldots \quad (5{\cdot}5)$$

Multiplying (5·2) by $\cos\alpha$ and (5·3) by $\sin\alpha$ and adding we get

$$\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = -\frac{\mu}{r^2}\cos\alpha\left[1 - \frac{1}{D}\left\{\frac{dr}{dt} - \frac{1}{D}\left(r\frac{d\theta}{dt}\right)^2\right\}\right]^3 \ldots (5{\cdot}6)$$

Hence $$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2}\left[1 - \frac{3}{D}\frac{dr}{dt} + \frac{3}{D^2}\left(r\frac{d\theta}{dt}\right)^2\right] \text{ nearly} \ldots (5{\cdot}61)$$

This can be written in the form

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} - \frac{3\mu}{D^2}\frac{1}{r^4}\left(r^2\frac{d\theta}{dt}\right)^2 + \frac{3\mu}{D}\cdot\frac{1}{r^2}\frac{dr}{dt}.$$

$$= -\frac{\mu}{r^2} - \frac{3\mu}{D^2}\frac{h^2(1+k\theta)^2}{r^4} + \frac{3\mu}{D}\frac{1}{r^2}\frac{dr}{dt} \ldots (5{\cdot}7)$$

2. The above equation can be transformed into (u, θ) form by putting $r = \frac{1}{u}$ and therefore

$$\frac{dr}{dt} = -h(1+k\theta)\frac{du}{d\theta}$$

and
$$\frac{d^2r}{dt^2} = -h^2u^2(1+k\theta\left[(1+k\theta)\frac{d^2u}{d\theta^2}+k\frac{du}{d\theta}\right]$$

Hence the equation (5·7) by substitution becomes

$$-h^2u^2(1+k\theta)\left[(1+k\theta)\frac{d^2u}{d\theta^2}+k\frac{du}{d\theta}\right]-h^2u^3(1+k\theta)^2$$

$$= \quad -(\mathrm{D}hk)u^2-\frac{3(\mathrm{D}hk)}{\mathrm{D}^2}h^2(1+k\theta)^2u^4-\frac{3(\mathrm{D}hk)}{\mathrm{D}}u^2h(1+k\theta)\frac{du}{d\theta}.$$

Therefore
$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2(1+k\theta)^2}+\frac{3\mu}{\mathrm{D}^2}u^2+\frac{2\mu}{\mathrm{D}h(1+k\theta)}\frac{du}{d\theta} \quad \ldots \quad (5·8)$$

This can be written in the form

$$\frac{d^2u}{d\theta^2}-2\frac{k}{(1+k\theta)}\frac{du}{d\theta}+u=\frac{\mu}{h^2(1+k\theta)^2}+\frac{3\mu}{\mathrm{D}^2}u^2$$

$$\text{where } k=\frac{\mu}{h\mathrm{D}} \quad \ldots \quad \ldots \quad (5·9)$$

3. In order to appreciate the relative importance of the various terms in the equation (5·8) it is convenient to give the calculated values of the coefficients for a planet, say Mercury.

$$\frac{\mu}{h^2}=\frac{\mathrm{G.\,M.}}{h^2}=\frac{6·67\times10^{-8}\times1·98\times10^{33}}{(2·76\times10^{19})^2}=1·73\times10^{-13}.$$

$$\frac{2\mu k}{h^2}\theta= \qquad 5·19\times10^{-17}\theta.$$

$$\frac{3\mu}{\mathrm{D}^2}u^2=\frac{13·2\times10^{25}\times3}{9\times10^{20}\times(·387\times1·49\times10^{13})^2}=1·32\times10^{-20}.$$

$$\frac{2\mu}{\mathrm{D}h(1+k\theta)}\frac{du}{d\theta}=\frac{2k}{(1+k\theta)}\frac{du}{d\theta}=-\frac{2\times1·5\times10^{-4}}{(2·76\times10^{19})(1+k\theta)^2}\frac{dr}{dt}$$

$$=-1·09\times10^{-23}\frac{dr}{dt}, \quad \text{(approximately)}.$$

For a whole revolution the total effect of the term containing $\frac{dr}{dt}$ is very small because it changes its sign.

4. The values of k for some of the other planets go on decreasing:—

Mercury	Venus	Earth	Mars
$1{\cdot}62771 \times 10^{-4}$	$1{\cdot}16531 \times 10^{-4}$	$\cdot 99124 \times 10^{-4}$	$\cdot 80645 \times 10^{-4}$

Section VI

THE FIRST APPROXIMATION

The first approximation can be obtained at once from the equation (5·8) by treating $k\theta$ as negligible, and ignoring the last two smallest terms. The equation then takes the form

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2} \qquad \dots \quad \dots \quad \dots \quad \dots \quad (6{\cdot}1)$$

which gives $u=\frac{\mu}{h^2}\left[1+e\cos(\theta-w)\right]$, a perfect ellipse. ... (6·2)

This is the well known Newton's form.

We also get

$$r^2\frac{d\theta}{dt}=h \qquad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (6{\cdot}3)$$

$$h^2=\mu l \qquad \text{where } l=\frac{b^2}{a} \qquad \dots \quad \dots \quad (6{\cdot}4)$$

$$v^2=\frac{2\mu}{r}-\frac{1}{a} \dots \qquad \dots \quad \dots \quad \dots \quad (6{\cdot}5)$$

$$T^2=\frac{4\pi^2}{\mu}a^3 \dots \qquad \dots \quad \dots \quad \dots \quad (6{\cdot}6)$$

Section VII

THE SECOND APPROXIMATION

The next approximation obviously is

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2(1+k\theta)^2} \qquad \dots \quad \dots \quad \dots \quad (7{\cdot}1)$$

which gives

$$u=\frac{\mu}{h^2(1+k\theta)^2}\left[1+e\cos(\theta-w)\right] \qquad \dots \quad \dots \quad (7{\cdot}2)$$

This is an intermediary form.

Section VIII

THE THIRD APPROXIMATION

1. The next approximation is obtained by neglecting $k\theta$ and omitting the last term which is the smallest in (5·8). This is

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}+\frac{3\mu}{c^2}u^2$$

where c has been substituted for D in order that it may take Einstein's form. Its second approximate solution is

$$u=\frac{\mu}{h^2}\left[1+e\cos(\theta-w)+\frac{3\mu^2}{c^2h^2}e.\,\theta.\,\sin(\theta-w)\right] \quad \ldots \quad \ldots \quad (8·2)$$

This can be put in the form

$$u=\frac{\mu}{h^2}\left[1+e\cos(\theta-w-\epsilon)\right]$$

$$\text{where} \quad \epsilon=\frac{3\mu^2}{c^2h^2}\theta. \quad \ldots \quad \ldots \quad (8·3)$$

2. It shows that the perihelion is advancing at the rate

$$\frac{\epsilon}{\theta}=\frac{3\mu^2}{c^2h^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8·4)$$

For one revolution, this gives

$$\epsilon=\frac{6\pi\mu^2}{c^2h^2}=\frac{24\pi^3a^2}{c^2T^2(1-e^2)} \quad \ldots \quad \ldots \quad \ldots \quad (8·5)$$

3. If the orbit be described by a particle of light, h becomes infinite in Relativity and so the equation becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{3\mu}{c^2}u^2=\frac{3\mu}{c^2R^2}\cos^2\theta \quad \ldots \quad \ldots \quad (8·6)$$

By successive approximations, this gives

$$u=\frac{\cos\theta}{R}+\frac{\mu}{c^2R^2}(\cos^2\theta+2\sin^2\theta) \quad \ldots \quad \ldots \quad (8·7)$$

Putting $x=r\cos\theta$ and $y=r\sin\theta$ we get

$$x=R-\frac{\mu}{c^2R}\frac{x^2+2y^2}{\sqrt{x^2+y^2}} \quad \ldots \quad \ldots \quad \ldots \quad (8·8)$$

Hence the angle between the asymptotes

$$x= \quad R- \frac{\mu}{c^2R} \; (\pm 2y)$$

is

$$\frac{4\mu}{c^2R} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8{\cdot}9)$$

This gives double the value for the deflection of light from a star passing close to the Sun, as compared to that given by the Newtonian Law.

Section IX

THE FOURTH APPROXIMATION

The equation (8·1) may approximately be written as :—

$$\frac{d^2u}{d\theta^2} + u - \frac{3\mu}{D^2} u^2 = \frac{\mu}{h^2} (1-2\,k\theta) \quad \ldots \quad \ldots \quad \ldots \quad (9{\cdot}1)$$

Consider

$$\frac{d^2u}{d\theta^2} + u - \frac{3\mu}{D^2} u^2 = o \ldots \quad \ldots \quad \ldots \quad (9{\cdot}2)$$

$$\text{Put } y = \frac{du}{d\theta} \text{ and therefore } \frac{d^2u}{d\theta^2} = y \frac{dy}{du}$$

Substituting these values in (9·1) and multiplying by $2\,\dfrac{dy}{du}$ and integrating we get:—

$$y^2 = \quad -u^2 + 2\,\frac{\mu}{D^2}\, u^3 + A, \text{ where A is a constant.}$$

Put $z = \quad u - \dfrac{D^2}{6\mu}$ and therefore $\dfrac{dz}{d\theta} = \dfrac{du}{d\theta}$ and multiply by $\dfrac{2D^2}{\mu}$.

$$\text{Then } 4\left(z+\frac{D^2}{6\mu}\right)^3 - \frac{2D^2}{\mu}\left(z + \frac{D^2}{6\mu}\right)^2 + \frac{2D^2}{\mu}A = \frac{2D^2}{\mu}y^2$$

$$\text{This gives} \qquad 4z^3 - \tfrac{1}{3}\,\frac{D^4}{\mu^2}\, z - \left(\tfrac{1}{27}\,\frac{D^4}{\mu^3} - \frac{2D^2}{\mu} A\right) = \frac{2D^2}{\mu}\, y^2.$$

But $y = \dfrac{du}{d\theta} = \dfrac{dz}{d\theta}$ and hence substituting for y, taking the square root and integrating we get

$$\sqrt{\left(\frac{\mu}{2D^2}\right)}.\; \theta = \int^{z} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} + \text{constant}$$

$$\text{where } g_2 = \tfrac{1}{3}\,\frac{D^4}{\mu^2} \qquad \text{and } g_3 = \left(\tfrac{1}{27}\,\frac{D^4}{\mu^3} - \frac{2D^2}{\mu}A\right)$$

But by the definition of $\wp$-function of Weierstrass

$$\int_x^\infty \frac{dx}{\sqrt{4x^3-g_2\,x-g_3}} = \wp^{-1}(x)$$

Hence $x = \wp\left\{\frac{1}{D}\sqrt{\frac{\mu}{2}}\,(\beta-\theta)\right\}$ where β is a constant.

Accordingly $u = \frac{D^2}{6\mu} + \wp\left\{\frac{\sqrt{\mu}}{D\sqrt{2}}\,(\beta-\theta)\right\}$ is a solution of (9·2).. (9·3)

Therefore the solution of (9·1) is

$$u = \frac{\mu}{h^2}(1-2k\theta) + \frac{D^2}{6\mu} + \wp\left\{\frac{1}{D}\sqrt{\frac{\mu}{2}}\,(\beta-\theta)\right\}$$

where $\wp(x)$ is known to be

$$= \frac{1}{x^2} + \sum_{m,n}\left\{\frac{1}{(x-2m\,\omega_1-2n\omega_2)^2} - \frac{1}{(2m\,\omega_1+2n\,\omega_2)^2}\right\} \quad . \quad . \quad (9\cdot4)$$

See Whittaker and Watson[13] : Modern Analysis, p. 434, and G. Prasad's Introduction to the Theory of Elliptic Functions and Higher Transcendentals.[14]

The solution can be expressed in series by successive approximations.

2. It is submitted that the preceding alternative method is simpler than those adopted by Prof. A. R. Forsyth in the Proceedings of the Royal Society of London[15] (Series A. Vol. 97, (1920) p. 145), Prof. F. Morley in the American Journal of Mathematics[16] (Vol. 43 (1921) p. 29), and Prof. James Pierpoint in the Bulletin of the American Mathematical Society[17] (1928) Vol 34 p., 582. These eminent mathematicians have taken the critical equation of Einsetin's as their starting point and obtained the solution in elliptic functions of $\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{c^2-1}{h^2} + \frac{2m}{h^2}\,u + 2m.u.^3$ ... (9·5)

If we take the form (9·1) and neglect k,

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} + \frac{3\mu}{D^2}\,u^2 \quad . \quad \ldots \quad \ldots \quad \ldots \quad (9\cdot6)$$

Multiplying by $\frac{2du}{d\theta}$, and integrating we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = A + \frac{2\mu}{h^2}\,u + \frac{2\mu}{D^2}\,u^3 \quad \ldots \quad \ldots \quad (9\cdot7)$$

where A is a constant determined by the initial conditions. This form

does not contain any apparent discrepancy as to the dimensions of its terms as (9·5) does.

The two equations are identical, but the solution of (9·6) by approximation, which has only four terms, is simpler than that of (9·5) which has five terms. As shown in the preceding paragraph the simplicity is not lost even when $\frac{\mu}{h^2}(1-2\,k\theta)$ is put in place of $\frac{\mu}{h^2}$.

3. If we multiply (9·2) by $2\,\frac{du}{d\theta}$ and integrate we get

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{D^2}\,u^3 - u^2 + B$$

Let α, β and γ be the three roots of $u^3 - \frac{D^2}{2\mu}u^2 + B = 0 = (u-\alpha)(u-\beta)(u-\gamma)$

$$= u^3 - (\alpha+\beta+\gamma)u^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)u - \alpha\beta\gamma$$

And so $$\alpha+\beta+\gamma = \frac{D^2}{2\mu}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = 0$$

$$\text{and } \alpha\beta\gamma = -B$$

From these we get $$\alpha+\beta-\frac{\beta}{\alpha\beta} = \frac{D^2}{2\mu}$$

and $$\alpha^2\beta^2 = (\alpha+\beta).\ B.$$

Thus when B is known the roots can be determined.

SECTION X

THE FIFTH APPROXIMATION

1. As the term $\frac{2\mu k}{h^2}\,\theta$ is comparable to $\frac{3\mu}{D^2}u^2$, the third approximation which gave Einstein's form after neglecting $\frac{2\mu k}{h^2}\,\theta$ was really not quite correct. The approximation should be:—

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2\,(1+k\theta)^2} + \frac{3\mu}{D^2}u^2 \qquad \dots \qquad \dots \qquad (10·1)$$

A rough solution of this can be obtained easily by putting $h(1+k\theta)=\mathrm{H}$ and treating the latter temporarily as a constant. For small values of $k\theta$, we get

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{\mathrm{H}^2}+\frac{3\mu}{\mathrm{D}^2}u^2$$

As in Section VIII, this gives

$$u=\frac{\mu}{\mathrm{H}^2}\left[1+e\cos(\theta-\omega-\varepsilon)\right] \quad \text{where } \varepsilon=\frac{3\mu^2}{\mathrm{D}^2\mathrm{H}^2}\theta$$

Re-instating h we get roughly

$$u=\frac{\mu}{h^2(1+k\theta)^2}\left[1+e\cos(\theta-\omega-\varepsilon)\right]$$

$$\text{where } \varepsilon=\frac{3\mu^2}{\mathrm{D}^2h^2(1+k\theta)^2} \qquad \ldots \quad (10{\cdot}2)$$

Hence so long as $k\theta$ remains small, the value of the advance of the perihelion is given by

$$\varepsilon=\frac{12\pi^2a^2}{c^2\mathrm{T}^2(1-e^2)}\,\frac{\theta}{(1+k\theta)^2} \qquad \ldots \quad \ldots \quad \ldots \quad (10{\cdot}3)$$

2. Similarly if we put $h(1+k\theta)$ in place of h we can obtain an approximate value for the advance of the perihelion from $(8{\cdot}4)$ and $(8{\cdot}5)$ for one revolution as

$$\varepsilon=\frac{3\mu^2}{\mathrm{D}^2h^2(1+2\pi k)^2} \qquad \ldots \quad \ldots \quad (10{\cdot}4)$$

$$=\frac{24\pi^3a^2}{\mathrm{D}^2\mathrm{T}^2(1-e^2)(1+2\pi k)^2} \qquad \ldots \quad (10{\cdot}5)$$

$$\text{where } k=\frac{\mathrm{GM}}{h.\mathrm{D}}$$

Section XI

THE SIXTH APPROXIMATION

The next approximation is the equation $(5{\cdot}9)$

$$\frac{d^2u}{d\theta^2}-\frac{2k}{(1+k\theta)}\frac{du}{d\theta}+u=\frac{\mu}{h^2(1+k\theta)^2}+\frac{3\mu}{\mathrm{D}^2}u^2.$$

(1) Consider first $$\frac{d^2u}{d\theta^2}-2k\frac{du}{d\theta}+u=0. \ldots \qquad \ldots \quad (11{\cdot}1)$$

Trying $u = A e^{\lambda\theta}$ we must have $\lambda^2 - 2k\lambda + 1 = 0$.

$\therefore \quad \lambda = k \pm i\sqrt{1-k^2}$ since $k < 1$.

Hence $u = A\ e^{k\theta} \left(e^{i\sqrt{1-k^2}\,\theta} + e^{-i\sqrt{-k^2}\,\theta}\right)$

$= 2A\ e^{k\theta} \cos\left(\sqrt{1-k^2}\ \theta\right)$

Similarly another solution is $u = 2Be^{k\theta} \sin\left(\sqrt{1-k^2}\ \theta\right)$
Hence neglecting k^2 the solution can be put in the form

$$u = \frac{\mu}{h^2}\ E\ e^{k\theta} \cos(\theta - \omega) \quad \ldots \quad \ldots \qquad (11\cdot2)$$

(2) By operating with $(D^2 - 2kD + 1)$ on $\frac{\mu}{h^2}(1 - 2k\theta)$

it is found that the solution of

$$\frac{d^2u}{d\theta^2} - 2k\frac{du}{d\theta} + u = \frac{\mu}{h^2}(1 - 2k\theta) \quad \ldots \qquad (11\cdot3)$$

is $$u = \frac{\mu}{h^2}(1 - 2k\theta) + \frac{\mu}{h^2}\ E\ e^{k\theta} \cos(\theta - \omega) \quad \ldots \quad \ldots \qquad (11\cdot4)$$

(3) Next consider

$$\frac{d^2u}{d\theta^2} - 2\frac{k}{1+k\theta}\frac{du}{d\theta} + u = 0 \quad \ldots \quad \ldots \qquad (11\cdot5)$$

Try $u = e^{k\theta(1-k\theta)} \cos(\theta - \omega)$, as $(1 - k\theta) = \frac{1}{1+k\theta}$ nearly.

It will be found that in the coefficients of the terms containing $\cos(\theta-\omega)$ and $\sin(\theta-\omega)$, neither any number nor k but only k^2 occurs, which is negligible. So the equation is satisfied approximately.

Hence a solution of (11·5) is

$$u = \frac{\mu}{h^2}\ E\ e^{\frac{k\theta}{1+k\theta}} \cos(\theta - \omega) \quad \ldots \qquad (11\cdot6)$$

which gives the complementary integral.

(4) By operating with $(D^2 - 2kD + 1)$ on $\frac{\mu}{h^2(1+k\theta)^2}$ and neglecting smaller terms, it will be found that an approximate solution of

$$\frac{d^2u}{d\theta^2} - 2\frac{k}{(1+k\theta)}\frac{du}{d\theta} + u = \frac{\mu}{h^2(1+k\theta)^2} \quad \ldots \qquad (11\cdot7)$$

is $$u = \frac{\mu}{h^2(1+k\theta)^2} + \frac{\mu}{h^2}\ E\ e^{\frac{k\theta}{1+k\theta}} \cos(\theta - \omega), \quad \ldots \qquad (11\cdot8)$$

so long as $k\theta$ does not become large.

(5) Substituting the approximate value (11·8) of u in the equation quoted above as the sixth approximation we get

$$\frac{d^2u}{d\theta^2}-\frac{2k}{1+k\theta}\frac{du}{d\theta}+u=\frac{\mu}{h^2(1+k\theta)^2}+\frac{3\mu}{D^2}\frac{\mu^2}{h^4(1+k\theta)^4}$$
$$+\frac{3\mu}{D^2}\frac{2\mu^2}{h^4}E\frac{e^{\frac{k\theta}{1+k\theta}}}{(1+k\theta)^2}\cos(\theta-\omega)$$
$$+\frac{3\mu}{D^2}\frac{\mu^2}{h^4}E^2e^{\frac{2k\theta}{1+k\theta}}\cos^2(\theta-\omega)$$

The second term on the right hand side is always too small as compared to the first and can be neglected. The fourth term is also very small compared to the third, and the period of $\cos^2(\theta-\omega)=\frac{1+\cos 2(\theta-\omega)}{2}$ does not correspond with that of $\cos(\theta-\omega)$; hence this also is negligible. Only the third term produces a continually increasing resonance. So that

$$\frac{d^2u}{d\theta^2}-\frac{2k}{1+k\theta}\frac{du}{d\theta}+u=\frac{\mu}{h^2(1+k\theta)^2}+\frac{6\mu^3E}{D^2h^4(1+k\theta)^2}e^{\frac{k\theta}{1+k\theta}}\cos(\theta-\omega) \quad (11·9)$$

(6) As $\dfrac{e^{\frac{k\theta}{1+k\theta}}}{(1+k\theta)}=\dfrac{1}{(1+k\theta)}$ nearly, it is convenient to consider

$$\frac{d^2u}{d\theta^2}-2k\frac{du}{d\theta}+u=A\frac{\cos\theta}{1+k\theta} \quad \cdots \quad \cdots \quad (11·10)$$

By trying $u=\frac{1}{2}A\dfrac{\theta\sin\theta}{1+k\theta}$ ... ... (11·11)

we find that:—

$$\frac{d^2u}{d\theta^2}-2k\frac{du}{d\theta}+u=A\frac{\cos\theta}{1+k\theta}-2A(\sin\theta+\theta\cos\theta)k.$$

If $A=\dfrac{6\mu^3E}{D^2h^4}$ and k is of the order 10^{-4}, the last term is negligible. Hence (11·11) is an approximate solution of (11·10).

(7) Now as a particular solution of (11·10) is (11·11), a term of the solution of (5·9) is known. Adding the term to the complementary integral, we obtain from (11·9) and (11·6).

$$u=\frac{\mu}{h^2(1+k\theta)^2}+\frac{\mu}{h^2}E\,e^{\frac{k\theta}{1+k\theta}}\cos(\theta-\omega)+\frac{3\mu^3}{D^2h^4}E\frac{e^{\frac{k\theta}{1+k\theta}}}{(1+k\theta)^2}\theta\sin(\theta-\omega)$$

approximately. ... (11·12)

This can be written in the form

$$u = \frac{\mu}{h^2(1+k\theta)^2}\left[1+\mathrm{E}\,(1+k\theta)^2\, e^{\frac{k\theta}{1+k\theta}} \cos\,(\theta-\omega-\varepsilon)\right]$$

$$\text{where } \varepsilon = \frac{3\mu^2}{\mathrm{D}^2 h^2}\;\frac{\theta}{(1+k\theta)^2} \qquad \cdots \qquad (11{\cdot}13)$$

$$= 3\left(\frac{k}{1+k\theta}\right)^2 \theta$$

(8) This solution can be verified by differentiation. When k^2 is neglected,

$$u = \frac{\mu}{h^2(1+k\theta)^2}\left[1+\mathrm{E}\,(1+k\theta)^2\, e^{\frac{k\theta}{1+k\theta}} \cos\,(\theta-\omega-\varepsilon)\right]$$

$$= \frac{\mu}{h^2}(1-2k\theta)\left[1+\mathrm{E}\,(1+2\,k\theta)\,(1+k\theta)\cos\,(\theta-\omega-\varepsilon)\right]$$

$$= \frac{\mu}{h^2}\left[1+\mathrm{E}\cos\,(\theta-\omega-\varepsilon)-2k\theta+\mathrm{E}\,k\theta\cos\,(\theta-\omega-\varepsilon)\right]$$

$$\therefore\ \frac{du}{d\theta} = \frac{\mu}{h^2}\Big[-\mathrm{E}\sin\,(\theta-\omega-\varepsilon)-2k+\mathrm{E}\,k\cos\,(\theta-\omega-\varepsilon)$$
$$-\mathrm{E}\,k\theta\sin\,(\theta-\omega-\varepsilon)\Big]$$

$$\therefore\ \frac{d^2u}{d\theta^2} = \frac{\mu}{h^2}\Big[-\mathrm{E}\cos\,(\theta-\omega-\varepsilon)-2\mathrm{E}\,k\sin\,(\theta-\omega-\varepsilon)$$
$$-\,\mathrm{E}\,k\theta\cos\,(\theta-\omega-\varepsilon)\Big]$$

Hence
$$\frac{d^2u}{d\theta^2}-2\,\frac{k}{1+k\theta}\,\frac{du}{d\theta}+u$$

$$= \frac{\mu}{h^2}\Big[-\mathrm{E}\cos\,(\theta-\omega-\varepsilon)-2\mathrm{E}\,k\sin\,(\theta-\omega-\varepsilon)-\mathrm{E}\,k\theta\cos\,(\theta-\omega-\varepsilon)$$
$$+1+\mathrm{E}\cos\,(\theta-\omega-\varepsilon)-2k\theta+\mathrm{E}\,k\theta\cos\,(\theta-\omega-\varepsilon)$$
$$+\,2k\,\mathrm{E}\sin\,(\theta-\omega-\varepsilon)\Big]$$

$$= \frac{\mu}{h^2}(1-2k\theta) = \frac{\mu}{h^2(1+k\theta)^2} \text{ (approximately).}$$

The solution is therefore true up to the order $\frac{\mu}{h^2}k = \frac{\mu^2}{h^3\mathrm{D}}$.

(9) It is easily seen that if we take

$$u = \frac{\mu}{h^2 (1+k\theta)^2}\left[1+E\,(1+k\theta)^n\, e^{\frac{k\theta}{1+k\theta}} \cos(\theta-\omega-\varepsilon)\right]$$

no other power will satisfy the differential equation except $n=2$.

Section XII

THE SEVENTH APPROXIMATION

In the new theory, the equations of motion are (5·7) and (5·41), *viz.*,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} - \frac{3\mu h^2}{D^2 r^4} + \frac{3\mu}{D}\frac{1}{r^2}\frac{dr}{dt} \text{ nearly}$$

$$\text{and } \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\mu}{D}\frac{1}{r}\frac{d\theta}{dt}.$$

From W. H. Besant's Dynamics,[18] Arts 160-161 we get the changes in the elements of Newton's ellipse, for which $r^2 \dfrac{d\theta}{dt} = h.$

1. *For a small radial force* $\delta a = f\,\delta t.\ 2e\sin\theta.\sqrt{\dfrac{a^3}{\mu(1-e^2)}}$

(1) If $f_1 = -\dfrac{3\mu h^2}{D^2}\dfrac{1}{r^4}$ nearly

$$\delta a_1 = -\frac{3\mu h^2}{D^2}\, 2e.\sqrt{\frac{a^3}{\mu(1-e^2)}}\ \frac{\sin\theta}{r^4}\, dt.$$

$$\text{But} \int_0^{2\pi} \frac{\sin\theta dt}{r^4} = \int_0^{2\pi} \frac{\sin\theta}{r^4}\,\frac{r^2 d\theta}{h} = \frac{1}{h}\int_0^{2\pi} \frac{\sin\theta\, d\theta}{r^2}$$

$$= \frac{1}{h}\int_0^{2\pi} \sin\theta.\frac{\mu^2}{h^4}\,(1+e\cos\theta)^2\, d\theta.$$

$$= \frac{\mu^2}{h^5}\int_0^{2\pi} \sin\theta\,(1+e\cos\theta)^2 d\theta = 0.$$

Hence for one revolution $\Delta a_1 = 0.$ ... ... (12·1)

(2) If $f_2 = \dfrac{3\mu}{D} \dfrac{1}{r^2} \dfrac{dr}{dt}.$ and $-\dfrac{dr}{r^2} = -\dfrac{\mu}{h^2} e \sin\theta \, d\theta.$

$$\delta a_2 = \frac{3\mu}{D} \cdot \frac{1}{r^2} \cdot \frac{dr}{dt} \cdot dt \cdot 2e \sin\theta . \sqrt{\frac{a^3}{\mu(1-e^2)}}$$

$$= \frac{e^2\mu^2}{Dh^2}\sqrt{\frac{a^3}{\mu(1-e^2)}} \sin^2\theta \, d\theta.$$

$$\text{But} \int_0^{2\pi} \sin^2\theta \, d\theta = \pi.$$

$$\therefore \text{ For one revolution } \Delta a_2 = \frac{6\pi e^2\mu^2}{Dh^2}\sqrt{\frac{a^3}{\mu(1-e)^2}}$$

$$= \frac{6\pi e^2\mu^2 a^2}{Dh^3} \qquad \dots \quad \dots \quad \dots \qquad (12{\cdot}2)$$

For a small transverse force $\delta a = f\,\delta t \, 2\sqrt{\dfrac{a^3}{\mu(1-e^2)}} \left\{1 + e\cos\theta\right\}$

(3) If $f_3 = \dfrac{\mu}{D} \dfrac{1}{r} \dfrac{d\theta}{dt}$

$$\delta a_3 = \frac{2\mu}{D}\sqrt{\frac{a^3}{\mu(1-e^2)}} \frac{\mu}{h^2} (1 + e\cos\theta)^2 \frac{d\theta}{dt} dt.$$

$$= \frac{2\mu^2}{Dh^3} a^2 (1 + 2e\cos\theta + e^2\cos^2\theta) \, d\theta.$$

$$\text{But} \int_0^{2\pi} (1 + 2e\cos\theta + e^2\cos^2\theta) \, d\theta = \pi(2 + e^2).$$

$$\therefore \quad \Delta a_3 = \frac{2\pi\mu^2 a^2 (2 + e^2)}{Dh^3} \qquad \dots \quad \dots \qquad (12{\cdot}3)$$

$$\text{Hence } \Delta a = \Delta a_1 + \Delta a_2 + \Delta a_3$$

$$= \frac{4\pi\mu^2 a^2}{Dh^3}(2e^2 + 1) \qquad \dots \quad \dots \quad \dots \qquad (12{\cdot}4)$$

2. *For a small radial force* $\delta e = f\,\delta t \sin\theta . \sqrt{\dfrac{a(1-e^2)}{\mu}}$

(1) If $f_1 = -\dfrac{3\mu h^2}{D^2} \dfrac{1}{r^4}$, then it is easily seen as in the preceding paragraph that

$$\Delta e_1 = 0 \qquad \dots \quad \dots \quad \dots \qquad (12{\cdot}5)$$

(2) If $f_2 = \dfrac{3\mu}{D}\dfrac{1}{r^2}\dfrac{dr}{dt}$, then as $-\dfrac{dr}{r^2} = -\dfrac{\mu}{h^2}\, e\sin\theta\, d\theta.$

$$\delta e_2 = \frac{3\mu}{Dr^2}\frac{dr}{dt}. \quad \sin\theta.\sqrt{\frac{a(1-e^2)}{\mu}}\,dt.$$

$$= \frac{3e\mu^2}{Dh^2}\sqrt{\frac{a(1-e^2)}{\mu}}\ \sin^2\theta.\ d\theta.$$

$$\text{But} \int_0^{2\pi} \sin^2\theta\, d\theta = \pi.$$

$$\therefore\quad \Delta e_2 = \frac{3\pi\, e\mu^2}{Dh^2}\sqrt{\frac{a(1-e^2)}{\mu}} = \frac{3\pi e\mu}{Dh} \qquad \ldots \qquad \ldots \qquad (12{\cdot}6)$$

For a small transverse force

$$\delta e = f\,\delta t\sqrt{\frac{a(1-e^2)}{\mu}}\left\{\cos\theta + \frac{e+\cos\theta}{1+e\cos\theta}\right\}$$

(3) If $f_3 = \dfrac{\mu}{D}\dfrac{1}{r}\dfrac{d\theta}{dt}$, then

$$\delta e_3 = \frac{\mu}{D}\sqrt{\frac{a(1-e^2)}{\mu}}\ \frac{\mu}{h^2}\left\{e+2\cos\theta + e\cos^2\theta\right\} d\theta.$$

$$\text{But} \int_0^{2\pi} (e+2\cos\theta + e\cos^2\theta)\, d\theta = 3\pi e.$$

$$\therefore\quad \Delta e_3 = 3\pi e.\ \frac{\mu^2}{Dh^2}\sqrt{\frac{a(1-e^2)}{\mu}} = 3\pi e\,\frac{\mu}{Dh} \qquad \ldots \qquad \ldots \qquad (12{\cdot}7)$$

Hence $\Delta e = \Delta e_1 + \Delta e_2 + \Delta e_3 = 6\pi e\,\dfrac{\mu}{Dh}$ $\qquad \ldots \qquad \ldots \qquad (12{\cdot}8)$

3. *For a small radial force* $\quad \delta\bar{\omega} = -f\,\delta t\,\dfrac{\cos\theta}{e}\sqrt{\dfrac{a(1-e^2)}{\mu}}$

(1) If $f_1 = -\dfrac{3\mu h^2}{D^2}\dfrac{1}{r^4}$

$$\delta\overline{\omega_1} = \frac{3\mu h^2}{D^2 e}\sqrt{\frac{a(1-e^2)}{\mu}}\ \frac{\cos\theta}{r^4}\,dt$$

$$= \frac{3\mu h^2}{D^2 e}\sqrt{\frac{a(1-e^2)}{\mu}}\ \frac{\cos\theta}{h}\ \frac{\mu^2}{h^4}\ (1+e\cos\theta)^2 d\theta.$$

$$\text{But} \int_0^{2\pi} \cos\theta\,(1+e\cos\theta)^2 = 2\pi e.$$

$$\therefore\ \Delta\overline{\omega_1} = \frac{3\mu h^2}{D^2 e}\sqrt{\frac{a(1-e^2)}{\mu}}\ \frac{\mu^2}{h^5}\ 2\pi e.$$

$$= \frac{6\pi\mu^3}{D^2 h^3}\sqrt{\frac{a(1-e^2)}{\mu}} = \frac{6\pi\mu^2}{D^2 h^2} \quad \ldots \quad (12.9)$$

(2) If $f_2 = \dfrac{3\mu}{D}\,\dfrac{1}{r^2}\,\dfrac{dr}{dt}$.

$$\delta\overline{\omega}_2 = -\frac{3\mu}{D}\,\frac{1}{r^2}\,\frac{dr}{dt}\cdot\frac{\cos\theta}{e}\sqrt{\frac{a(1-e^2)}{\mu}}\,dt.$$

$$= -\frac{3\mu}{D}\cdot\frac{\mu}{h^2}\cdot e\sin\theta\,d\theta.\,\frac{\cos\theta}{e}\sqrt{\frac{a(1-e^2)}{\mu}}$$

$$\text{But} \int_0^{2\pi} \sin\theta\cos\theta\,d\theta. = 0.$$

$$\therefore\ \Delta\overline{\omega}_2 = 0. \quad \ldots \quad \ldots \quad \ldots \quad (12.10)$$

For a small transverse force

$$\delta\overline{\omega} = f\,\delta t.\,\frac{\sin\theta}{e}\sqrt{\frac{a(1-e^2)}{\mu}}\left(\frac{2+e\cos\theta}{1+e\cos\theta}\right)$$

(3) If $f_3 = \dfrac{\mu}{D}\,\dfrac{1}{r}\,\dfrac{d\theta}{dt}$

$$\delta\overline{\omega}_3 = \frac{\mu}{D}\,\frac{1}{r}\,\frac{d\theta}{dt}\cdot\frac{\sin\theta}{e}\sqrt{\frac{a(1-e^2)}{\mu}}\left(\frac{2+e\cos\theta}{1+e\cos\theta}\right)dt.$$

$$= \frac{\mu}{eD}\sqrt{\frac{a(1-e^2)}{\mu}}\ \frac{\mu}{h^2}\,(2+e\cos\theta)\sin\theta\ d\theta.$$

$$\text{But} \int_0^{2\pi} \sin\theta\,(2+e\cos\theta)\,d\theta = 0$$

$$\therefore\ \Delta\overline{\omega}_3 = 0. \quad \ldots \quad \ldots \quad \ldots \quad (12.11)$$

Hence $\Delta\overline{\omega} = \Delta\overline{\omega}_1 + \Delta\overline{\omega}_2 + \Delta\overline{\omega}_3$

$$= \frac{6\pi\mu^2}{D^2 h^2} \text{ (for one revolution)} \quad (12.12)$$

From (8·5) this $= \dfrac{24\pi^3 a^2}{D^2 T^2 (1-e^2)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12.13)$

4. (1) The orbit of the planet which is actually observed has the elements

$$a' = (a + \triangle a)$$
$$\text{and } e' = (e + \triangle e)$$

These therefore show no variations.

(2) But the rotation of the perihelion is given by

$$\triangle \bar{\omega} = \frac{6\pi\mu^2}{D^2 h^2} \text{ which is observed.}$$

5. The other terms in (5·61) which have been neglected produce no appreciable effect on the advance of the perihelion $\triangle\omega$, for three of the terms are zero, and two of them are of the order 10^{-17} and 10^{-15} respectively, which are both negligible. Similarly the effect of the other terms in (5·4) which have been neglected is negligible.

The method of treating the disturbed elliptic motion as being due to small additional forces which has been followed in this Section is not known to have been adopted previously.

CHAPTER II

Applications of the New Theory

SECTION I

THE ADVANCE OF THE PERIHELION

1. From the equation (11·13) the advance of the perihelion is given by

$$\varepsilon = \frac{3\mu^2}{D^2 h^2} \frac{\theta}{(1+k\theta)^2} \text{ where } k = \frac{G.\ M.}{h.\ D.} \text{ and}$$

for one revolution this gives:—

$$\varepsilon = \frac{6\pi\mu^2}{D^2 h^2} \frac{1}{(1+2\pi k)^2}, \ k\theta \text{ being small.} \qquad \ldots \qquad \ldots \qquad (13·1)$$

An equivalent formula from (8·5) is

$$\varepsilon = \frac{24\ \pi^3 a^3}{c^2 T^2 (1-e^2)} \frac{1}{(1+2\pi k)^2} \text{ for one revolution.} \quad \ldots \qquad (13·2)$$

Leverrier in 1859 and Newcomb in 1895 calculated the advance of the perihelion. The observed advance of the orbit is about 574″ per century, and the calculated perturbations produced by all the known planets, on certain assumptions amount to about 532″, leaving an excess of 42″ per century. But the mean value[12] comes to 40″·00 ± 1″·16. The method of calculating the perturbations will be examined in Chapter III.

SECTION II

THE DEFLECTION OF LIGHT

1. If light be regarded as a material particle its motion will be governed by the same equations. But for light h and therefore $h(1+k\theta)$ is very large. Hence the equation (5·8) reduces to:—

$$\frac{d^2u}{d\theta^2}+u = \frac{3\mu}{D^2}u^2 \text{ as a first approximation.} \quad \ldots \quad (14{\cdot}1)$$

This is the same as (8·6)

Accordingly, as in Section VIII, the rays of light from a star when passing close to the Sun should as a first approximation be deflected by an angle $\frac{4\mu}{R}$ which is double of that given by Newton's law.

2. The value on Newton's Theory corresponds to		. $0''{\cdot}87$
„ „ „ Einstein's „ „		. $1''{\cdot}745$
The observed value at Sobral[19] (1919) was		. $1''{\cdot}98 \pm 0''{\cdot}12$
The corrected value for Sobral (Hopmann[20]) is		. $2''{\cdot}16$
The American Expedition[21] (1922) after correction is		$2''{\cdot}30$
The observed values at Takengon, North Sumatra (1929) are		
(Freundlich and Klüber)[22]	minimum	. $1''{\cdot}97$
	maximum	. $2''{\cdot}87$
	mean	. $2''{\cdot}24 \pm 0''{\cdot}10$
The corrected mean value for North Sumatra (Freundlich)[23] is		. $2''{\cdot}20$
The value with the probable errors (Lanczos[24]) is		. $2''{\cdot}20 \pm 0''{\cdot}10$

The results of the above expeditions, including some others, and the effect of the corrections will be more fully discussed in Chapter III.

3. (i) In Relativity the fundamental equation is

$$ds^2=dx^2+dy^2+dz^2+c^2\ (\sqrt{-1}\ dt)^2 \quad \ldots \quad \ldots \quad (14{\cdot}2)$$

Hence for a body moving with the speed of light $ds=0$. Accordingly

$$h= r\frac{d\phi}{ds} = \infty \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14{\cdot}3)$$

Thus there is no option but to assume h to be infinite. But if a ray of light from a star passes round the sun, $r\,\frac{d\theta}{dt}$ can never exceed the tangential velocity c; and so $h= r^2\,\frac{d\theta}{dt}$ can never exceed $r.\,c.$ When the arc of the hyperbolic path near the sun is almost circular $h=r.\,c.$ Both r and c being finite, h cannot possibly be infinite. Hence in equation (8·1) it is wrong

to neglect $\frac{\mu}{h^2}$ and treat it as zero. The term $\frac{2\mu}{Dh}\frac{du}{d\theta}$, which equals $-\frac{2\mu}{Dh^2}\frac{dr}{dt}$ is certainly very small, particularly when in a nearly circular arc $\frac{dr}{dt}$ is nearly zero and also changes sign. This may be neglected; but the remaining two terms are of the same order.

As h is not greater than $r.\ c$

$$\frac{\mu}{h^2} \text{ is not less than } \frac{\mu}{c^2}u^2$$

$$\text{Hence} \quad \frac{d^2u}{d\theta^2}+u=\frac{\mu}{c^2}u^2+\frac{3\mu}{D^2}u^2 \text{ at least} \quad \dots \quad \dots \quad (14{\cdot}4)$$

$$=\frac{4\mu}{c^2}u^2 \text{ where } D=c \quad \dots \quad \dots \quad (14{\cdot}5)$$

Comparing this equation with (8·6), it is at once seen that the real deflection is nearly $\frac{8}{3}$ times the value of Einstein's, which is twice that of Newton's.

Accordingly the angle of deflection is

$$= \tfrac{8}{3}\times 0''{\cdot}87 \doteq 2{\cdot}66 \text{ times that of Newton's}$$

$$= 2''{\cdot}32 \quad \dots \quad \dots \quad \dots \quad \dots \quad (14{\cdot}6)$$

This accords with the observed values quoted in the preceding paragraph. It is submitted that as the large excess over Einstein's value destroys the fundamental assumption in Relativity that ds=0, the very observation of the deflection of light which was at one time believed to have verified Relativity must now be taken to disprove it.

4. The effect of refraction through the corona of the sun, treated as a medium of concentric spheres, with density decreasing as a function of the distance from the centre, is very small if the index of refraction is small.

(i) As $\mu \sin\phi=(\mu+d\mu)\sin(\phi-d\epsilon)=\mu\sin\phi-\mu\cos\phi.d\epsilon+d\mu\sin\phi$

$$\therefore\ d\mu.\sin\phi=\mu.\cos\phi\ d\epsilon. \quad \text{or} \quad \frac{d\mu}{\mu}=\frac{d\epsilon}{\tan\phi}$$

Hence if at the surface of the sun the values be $\mu=\mu_0$, $r=a$ and $\phi=\alpha$, then

$$\frac{d\epsilon}{d\mu}=\frac{1}{\mu}.\frac{\mu_0\ a.\ \sin\alpha}{\sqrt{\mu^2r^2-\mu_0{}^2a^2\sin^2\alpha}}$$

$$\therefore\ \varepsilon = \mu_0\, a\sin\alpha\int_0^{\mu_0}\frac{d\mu}{\mu\sqrt{\mu^2r^2-\mu_0{}^2a^2\sin^2\alpha}} \quad \dots \quad \dots \quad (14{\cdot}7)$$

(ii) Also as $r\mu \sin\phi = r\mu' \sin\phi'$, $\mu p = \mu' p' = h$.

Hence $$\mu^2 = \frac{h^2}{p^2} = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

When the ray is approaching the sun, $\frac{du}{d\theta}$ is positive.

Therefore $\theta = h\int^{u} \frac{du}{\sqrt{\mu^2 - h^2u^2}}$ gives the equation for the path of the ray. ... ... (14·8)

SECTION III

THE SHIFT OF THE FRAUNHOFER LINES

1. In the New Theory, light is regarded as a mere particle of matter called "radion." If a radion comes straight from the Sun to the Earth, then in (5·7) $\frac{d\theta}{dt} = 0$. So there will be no aberration.

Further as the velocities of gravitons and radions are about the same, there will be no effect of the Doppler principle. A radion will travel from the Sun to the Earth, surrounded by gravitons travelling with an equal velocity. The only effect will be that gravitons will be expanding in spherical surfaces, and therefore their intensity round about the radion will gradually diminish as the inverse square of the distance.

The equation will simply be $\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$. During the path, the radion will be attracted both by the Sun and the Earth. If M and m be the masses, a and b the radii of the Sun and the Earth and d the distance between them, then measuring r from the centre of the Sun,

$$V^2 = V_0^2 + 2G\int\left[-\frac{M}{r^2} + \frac{m}{(d-r)^2}\right]dr$$

$$= V_0^2 + 2G\left[\frac{M}{r} + \frac{m}{(d-r)}\right] \quad \dots \quad \dots \quad (15·1)$$

(i) If c be the velocity of the radion at ∞, then $V_0 = c$

At the Sun $V_s^2 = c^2 + 2G\left(\frac{M}{a} + \frac{m}{d-a}\right)$

At the Earth $V_e^2 = c^2 + 2G\left(\frac{M}{d-b} + \frac{m}{b}\right)$

Now a particle starting from the Sun with velocity V_s will have its velocity reduced at the Earth to V_e. It would, therefore, seem at the Earth as if the solar atoms had lower frequencies, *i.e.*, greater periods in the ratio

$$\frac{T_s}{T_e} = \left(\frac{V_s^{\,2}}{V_e^{\,2}}\right)^{\frac{1}{2}} = \frac{1+\frac{G}{c^2}\left(\frac{M}{a}+\frac{m}{d-a}\right)}{1+\frac{G}{c^2}\left(\frac{M}{d-b}+\frac{m}{b}\right)} \text{ nearly} \quad \dots \quad (15\text{·}2)$$

(ii) Or if the velocity at the Sun be taken to be c then $V_s = c$ and

$$V_0^{\,2} = C^2 - 2G\left(\frac{M}{a}+\frac{m}{d-a}\right)$$

$$\therefore \frac{T_s}{T_e} = \frac{V_s}{V_e} = \frac{1}{\left[1-2\frac{G}{c^2}\left(\frac{M}{a}+\frac{m}{d-a}\right)+2\frac{G}{c^2}\left(\frac{M}{d-b}+\frac{m}{b}\right)\right]^{\frac{1}{2}}}$$

$$= \frac{1+\frac{G}{c^2}\left(\frac{M}{a}+\frac{M}{d-a}\right)}{1+\frac{G}{c^2}\left(\frac{M}{d-b}+\frac{m}{b}\right)} \text{ nearly} \quad \dots \quad (15\text{·}2)$$

(iii) The equation (15·2) reduces approximately to

$$\frac{T_s}{T_e} = \frac{1+\frac{G}{c^2}\cdot\frac{M}{a}}{1+\frac{G}{c^2}\cdot\frac{m}{b}} \text{ as } d \text{ is large} \quad \dots \quad (15\text{·}3)$$

This is the formula deduced by Einstein in Relativity. He puts it in astronomical units as:—

$$\frac{1+\frac{M}{R}}{1+\frac{m}{r}}$$

2. (i) The calculated values are as follows:—

$$\frac{G}{c^2}\,\frac{m}{b} \text{ for the Earth} = \frac{5\text{·}974 \times 10^{27} \times 6\text{·}658 \times 10^{-8}}{(2\text{·}9986)^2 \times 10^{20} \times 6371 \times 10^5}$$

$$= \text{·}000{,}000{,}000{,}69$$

$$\frac{G}{c^2}\,\frac{m}{a} \text{ for the Sun} = \frac{1\text{·}983 \times 10^{33} \times 6\text{·}658 \times 10^{-8}}{(2\text{·}9986)^2 \times 10^{20} \times 6955 \times 10^7}$$

$$= \text{·}00000211$$

$\frac{G}{c^2}\ \frac{M}{d}$ for the Sun and the Earth

$$= \frac{1{\cdot}983 \times 10^{33} \times 6{\cdot}658 \times 10^{-8}}{(2{\cdot}9986)^2 \times 10^{20} \times 14945 \times 10^9}$$

$$= {\cdot}000{,}000{,}0098$$

Accordingly, the corrections provided by New Relativity are not appreciably large, and the value of the ratio is the same as Einstein's and

$$=1{\cdot}000002111 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15{\cdot}4)$$

The displacement of the blue light ($\lambda=4000$ Å. U.) is

$$4000 \times {\cdot}000002111 = {\cdot}008444.\ \text{Å.U.}$$

(ii) The observed value as given by St. John[19] (1917) = ·0036. Å.U.

(iii) St. John also once found the displacement to be ·000 for cyanogen lines. The possibility for such a small value, as also the values obtained by Evershed (1914, 1918), Schwarzschild (1914) and Grebe and Bachem (1919), will be discussed in Chapter III.

3. It may here be pointed out that as the theoretical value tallies approximately with the observed value, it disproves the assumption in Relativity that the velocity of light in a gravitational field remains constant. A change in the medium of course alters the velocity; but it now appears that a change in the gravitational potential also is effective.

4. In Relativity the formula for the ratio of vibrations is obtained from the equation $ds^2 = -\frac{1}{\gamma}\,dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta.\ d\phi^2 + \gamma dt^2$ (where a particular solution gives $\gamma = 1 - \frac{2m}{r}$) by treating the atoms at rest and therefore putting $dr = d\theta = d\phi = 0$, which give $ds^2 = \gamma.\ dt^2$. or $\frac{dt}{ds} = \frac{1}{\sqrt{\gamma}} = \left(1 + \frac{m}{r}\right)$ nearly.

It is submitted that when absolute motion is denied, the assumption of an absolute rest is illogical.

SECTION IV

THE COMPANION OF SIRIUS

1. For the companion of Sirius, the theoretical value first tallied with the observed value.

For the companion (α canis maj.' B.), the ratio

$$\frac{M_2}{a_2} = \frac{{\cdot}96 \text{ of sun's mass}}{{\cdot}034 \text{ of sun's radius}}$$

$$= 28 \text{ times the ratio for the sun.}$$

The displacement of the Fraunhofer lines should therefore be about 28 times.

The observed displacement is about 30 times.

2. But Vyssotshiy[10] has recently found that the density of the companion is much less, and therefore its radius much larger.

If this be right then the observed value would be about half as much as the theoretical value, giving the same discrepancy as in the case of the sun.

3. It is easy to see that the gravitational effect of Sirius A on the light from Sirius B would be negligible. Let their masses be M_1 and M_2, radii be a_1 and a_2, the distance between them be p and their distance from the Earth be d.

Let r be the distance of a radion from S_B and R its distance from S_A and let θ be the angle between r and R.

$$\text{Then } V^2 = V_0^2 + 2\int -\frac{GM_2}{r^2}\,dr + 2\int -\frac{GM_1}{R^2}\cos\theta\,dr.$$

$$= V_0^2 + 2\frac{GM_2}{r} + \frac{GM_1}{p}\left(\theta + \frac{\sin^2\theta}{2}\right) \quad \ldots \quad \ldots \quad (16{\cdot}1)$$

If $V=c$ at S_B where $r=a_2$ and $\theta=\frac{\pi}{2}$, then $V_0^2 = c^2 - 2\frac{GM_2}{a_2} - \frac{GM_1\pi}{2p}$.

Also when $r= d-a=d$ nearly and $\theta=0$ nearly, we have

$$V_e^2 = c^2 - \frac{2GM_2}{a_2} - \frac{GM_1\pi}{2p} + \frac{2GM_2}{d}.$$

$$\text{Hence} \qquad \frac{V_s}{V_e} = 1 + \frac{G}{c^2}\left[\frac{M_2}{a_2} + \frac{M_1\pi}{4p}\right] \quad \ldots \quad \ldots \quad (16{\cdot}2)$$

It is thus apparent that p being very large in comparison with a_2, the effect of the second term is almost nil.

Section V

THE PLANETS AND THE STARS

On the analogy of (15·2) the formulæ for the superior and inferior planets can be put down at once.

1. (i) When Jupiter is behind the Sun, and M' is its mass and d_1 its distance from the Sun,

$$\frac{T_s}{T_e} = \frac{1 + \frac{M}{a} + \frac{M'}{a+d_1} + \frac{m}{d-a}}{1 + \frac{M}{d-b} + \frac{M'}{d+d_1-b} + \frac{m}{b}} \quad \ldots \quad \ldots \quad \ldots \quad (17{\cdot}1)$$

(ii) When Jupiter is behind the Earth

$$\frac{T_s}{T_e} = \frac{1+\dfrac{M}{a}+\dfrac{M'}{d_1-a}+\dfrac{m}{d-a}}{1+\dfrac{M}{d-b}+\dfrac{M'}{d_1-d+b}+\dfrac{m}{b}} \qquad \cdots \quad \cdots \quad (17{\cdot}2)$$

2. (i) When Venus (m_2, d_2) is behind the Sun

$$\frac{T_s}{T_e} = \frac{1+\dfrac{M}{a}+\dfrac{m_2}{a+d_2}+\dfrac{m}{d-a}}{1+\dfrac{M}{d-b}+\dfrac{m_2}{d+d_2-b}+\dfrac{m}{b}} \qquad \cdots \quad \cdots \quad (17{\cdot}3)$$

(ii) When Venus is between the Sun and the Earth, the effect of Venus is practically nullified as light from the Sun to the Earth approaches to and recedes from Venus.

3. Unfortunately, the ratios for the planets are too small as compared to that for the Sun, and the more accurate formulæ cannot give any better results at present.

4. Similarly the ratio for an ordinary star or nebula is too small. For example, the value of $\dfrac{GM}{c^2 a}$ for M.31 (in Andromeda) comes to ·0000000238 which is negligible.

5. Another White Dwarf (40 Eridani B) has the ratio

$$\frac{M'}{a'} = \frac{{\cdot}44M}{{\cdot}019a} = 23{\cdot}1\,\frac{M}{a}.$$

Its displacement is therefore large enough and the measurement when made will furnish another test.

Section VI

EXPERIMENTS RECONCILED

1. **Airy** and **Hoek** found that the direction in which a star is seen remains unaltered, whether the telescope be filled with air or water. In the wave theory of light this implies that the ether waves inside matter must be *partially* carried along by the moving matter with a velocity diminished in the ratio $\left(1-\dfrac{1}{\mu^2}\right)$, where μ is the index of refraction. On the other hand, **Michelson** and **Morley's** experiment on the interference fringes with monochromatic light showed that it makes no difference

whether light travels backwards and forwards along the direction of the Earth or in a direction perpendicular to it, which implies that ether just outside a moving matter is *wholly* carried along with the velocity of the moving matter. The two results are irreconcilable.

2. But if a light particle from a star is arriving at a telescope with velocity c and finds the telescope moving horizontally with a velocity u, then the resultant effect is the same as if the telescope were reduced to rest, and an equal and opposite velocity $-u$ be added to the velocity of light. By tilting the telescope appropriately, the resultant path of light can be made to lie along the axis of the telescope. When the telescope is relatively at rest the resultant path cannot change its direction when the telescope is filled with water instead of air, only the relative velocity of light along the axis is decreased in water. The decrease of the velocity in water is the same both vertically and horizontally; hence the resultant path in spite of the reduced velocity remains unchanged.

3. (1) In Michelson and Morley's experiment monochromatic light produced on the earth has to be used. Like ordinary particles of matter it already possesses the velocity of the earth in addition to its own velocity c. This has been overlooked by the experimenters.

According to Newton's law $c+v \neq c-v \neq c$
Einstein gives to c the properties of infinity although c is finite. He arbitrarily assumes that $c+v=c$ and $c-v=c$. It is by this strange hypothesis that he tries to explain the result obtained by Michelson and Morley, but his is not an explanation but a mere assumption.

The real explanation is that Michelson and Morley misinterpreted their experiment. What they thought to be c was really $(c-v)$ in one direction and $(c+v)$ in the opposite direction, and so

$$\frac{1}{(c-v)+v}+\frac{1}{(c+v)-v} \equiv \frac{1}{c}+\frac{1}{c}.$$

"The corpuscular theory had implied a different mode of travel . . if it travelled like particles shot out from a gun, then its speed of travel would be always the same *relative to the gun from which it was fired.* "Does light travel like waves or like particles?" When the question is framed in this way, the Michelson-Morley experiments unambiguously support the latter alternative". (James Jeans)[25].

4. The stumbling block in the way of the corpuscular theory of light has been the phenomena of interference and diffraction. In my Unified Theory of Physical Phenomena it has been shown that all the

phenomena including interference and diffraction can be easily explained on a Rotational Theory of light. If a beam of monochormatic light consists of a swarm of material particles, either rotating round their path of longitudinal motion or even spinning round their own axes with one uniform period, a permanent difference in the phase of the resultant effect will be obtained, if the beam be split up into two parts, one of which travels along a longer distance than the other. When the two are made to rejoin, successive maximum and minimum effects will be produced, which would be periodic, and will necessarily cause the phenomena of interference. Rotating radions will thus possess both the particle and the wave aspects simultaneously, reconciling the phenomena of diffraction and scintillation.

SECTION VII

THE CHARACTERISTICS OF THE ORBIT

The equations are :—

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-\frac{\mu}{r^2}-\frac{3\mu}{D^2}\,\frac{h^2(1+k\theta)^2}{r^4}+\frac{3\mu}{D}\,\frac{1}{r^2}\,\frac{dr}{dt}\quad\ldots\quad(5\cdot7)$$

$$\frac{d^2u}{d\theta^2}-2\frac{k}{1+k\theta}\,\frac{du}{d\theta}+u=\frac{\mu}{h^2(1+k\theta)^2}+\frac{3\mu}{D^2}\,u^2\quad\ldots\quad\ldots\quad(5\cdot9)$$

and $$u=\frac{\mu}{h^2(1+k\theta)^2}\left[1+E(1+k\theta)^2\,e^{\frac{k\theta}{1+k\theta}}\cos(\theta-w-\varepsilon)\right]\text{ nearly.}$$

where $$\varepsilon=\frac{3\mu^2}{D^2h^2}\,\frac{\theta}{(1+k\theta)^2}\quad\ldots\quad\ldots\quad\ldots\quad(11\cdot13)$$

1. If $\frac{1}{r}=\alpha+\beta\cos\theta$ for an ellipse, then

$$\frac{1}{SA}=\alpha+\beta\text{ and }\frac{1}{SA'}=\alpha-\beta.$$

From these $a=\frac{\alpha}{\alpha^2-\beta^2}$, $b=\frac{1}{\sqrt{\alpha^2-\beta^2}}$ and $e=\frac{\beta}{\alpha}$

Hence if (11·13) be treated as resembling an ellipse, then roughly speaking

(*i*) its semi-major axis $= \dfrac{1}{\mu} h^2(1+k\theta)^2$ as observed, ... (18·1)

(*ii*) its semi-minor axis $= \dfrac{h^4}{\mu^2} \quad \dfrac{1}{1-E^2-4k\theta}$ as observed, (18·2)

and (*iii*) its eccentricity $= E \quad e^{\frac{k\theta}{1+k\theta}} (1+k\theta)^{\,2}$ as observed, (18·3)

2. These suggest that the orbit has the following characteristics in contrast with the Newtonian orbit:

(*a*) It rotates round the Sun at a measurable speed.
(*b*) Its major axis has a tendency to increase very slowly.
(*c*) Its minor axis also has a tendency to increase, but slightly less slowly.
(*d*) Its eccentricity has a tendency to increase slowly.
(*e*) The orbit, therefore, tends to enlarge in size, but becomes slightly elongated.
(*f*) The orbit instead of becoming more and more circular tends to approach a parabola.
(*g*) Planets are tending to resemble comets, until the parabolic path is reached when they would leave the solar system.

3. From (5·7) it is apparent that the last term changes its sign, while the others do not, as the planet approaches to or recedes from the Sun. At corresponding points in the orbit, the other terms have equal magnitudes and are directed towards the Sun. But the last term which contains $\dfrac{dr}{dt}$ has an additive effect for half the orbit and a subtractive effect for the other half. The net result is a gain of attraction towards the Sun. For ordinary planets (except perhaps Mercury) the orbits are nearly circular and $\dfrac{dr}{dt}$ is very small. It is small even for Mercury, but the term is appreciable in the case of comets and an increased velocity can be noticeable.

4. The value for the changes in the elements have been obtained in (12·5), (12·8) and (12·12). They also show that:—

(*i*) $\triangle a$ has a tendency to increase,
(*ii*) $\triangle e$ has a tendency to increase,
and that (*iii*) $\triangle \omega$ is increasing at a measurable rate.

These give the approximate results. But we must remember that what we observe are $T'=T+\triangle T$, $a'=a+\triangle a$, and $e'=e+\triangle e$. These therefore hardly show any variations. On the other hand $\triangle\omega$ is actually observed as increasing slowly.

Chapters III and IV

1. Chapter III will deal more fully with the values for (a) the deflection of light, (b) the displacement of the Fraunhofer lines, for (i) the sun, and (ii) the companion of Sirius, and (c) the advance of the perihelia of the inner planets.

2. The apparent velocities of approach of some of the nearest nebulæ can be considerably reduced by taking into account the galactic motion, but they do not *wholly* disappear, and even if they disappear the corrections cannot produce velocities of recession proportional to their distances. Relativity completely fails in this respect. See Prof. J. H. Reynold's[26] article in Nature, Vol. CXXX, July to December (1932), pp. 458—462.

It will be shown in Chapter IV how under the equations, Nebulæ, without any force of cosmic repulsion, can have velocities proportional to distances both of recession and approach, and how the equations lead to a stable and not an exploding universe.

3. Fizeaus' experiment will become easily intelligible and so would De Sitter's conclusion from the test of Binary stars that the velocity of light is independent of the velocity of the source in the direction of its motion (though not transversely).

4. An attempt will be made in Part II to show that the Universe is cyclic.

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A THEOREM CONCERNING THE ZEROS OF THE LAPLACE-ABEL INTEGRAL

BY S. P. JAIN

MATHEMATICS DEPARTMENT, ALLAHABAD UNIVERSITY

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In this note I propose to prove a theorem for the Laplace-Abel integral analogous to Theorem 6 of the Cambridge Tract No. 18 for the Dirichlet's series. The theorem in view may be stated as follows :—

Theorem :—

If

(1) *$\varphi(z)$ is an analytic function of z $(=x+iy)$ in the interior of the strip $|y| \leq k$, $x \geq -\delta$, where k and δ are some positive numbers, and satisfies the inequality*

$$\varphi(z) = O\left(e^{Mx}\right)$$

uniformly throughout this strip as $|z| \to \infty$;

(2) $f(s) = \int_0^\infty \varphi(x) e^{-sx} dx$, *which is certainly an analytic function of s in the half plane $\sigma \geq M+\delta_1 > M$, $\delta_1 > 0$, possesses an infinity of zeros in this half plane ;*

then $f(s)$ is identically zero.

Suppose $s_1, s_2, s_3 \ldots s_n$ are the zeros of $f(s)$. Then, since the zeros of an analytic function are isolated, $s_n \to \infty$ as $n \to \infty$ in the half-plane $\sigma \geq M + \delta_1 > M$.

Take a point $P(x)$ on the real axis in the z-plane. If x be sufficiently large, then we can describe a circle C of radius $e^{-\epsilon x}$ round P such that C lies wholly within the strip defined in (1), ϵ being an arbitrarily small positive number. By Cauchy's theorem, we shall have then

$$\varphi^{\mu}(x) = \frac{\mu!}{2\pi i} \int_{c} \frac{\varphi(z)dz}{(z-x)^{\mu+1}}$$

so that

$$|\varphi^{\mu}(x)| < K\mu!\, e^{\mu\epsilon x} e^{M\left(x+e^{-\epsilon x}\right)}$$

i.e.,

(3) $\varphi^{\mu}(x) = O\left(e^{(M+\epsilon\mu)x}\right)$, for $\mu = 0, 1, 2, 3 \ldots\ldots\ldots$

Now consider

$$sf(s) = s\int_0^\infty \varphi(x)\, e^{-sx}\, dx,$$

the integral being absolutely and uniformly convergent throughout the half-plane $\sigma \geq M + \delta_1 > M$.

Integrating by parts, we have if $\sigma \geq M + \delta_1 > M$,

$$(4) \qquad sf(s) = \Big\{-\varphi(x)\, e^{-sx}\Big\}_0^\infty + \int_0^\infty \varphi'(x)\, e^{-sx}\, dx$$

$$= \varphi(0) + \int_0^\infty \varphi'(x\; e^{-sx}\, dx$$

$$= \varphi(0) + \left\{-\frac{\varphi'(x)\, e^{-sx}}{s}\right\}_0^\infty + \frac{1}{s}\int_0^\infty \varphi''(x)\, e^{-sx}\, dx,$$

$$= \varphi(0) + \frac{\varphi'(0)}{s} + \frac{1}{s}\int_0^\infty \varphi''(x)\, e^{-sx}\, dx,$$

[by virtue of (3)]

Now the integral $\int_0^\infty \varphi''(x)\, e^{-sx}\, dx$, being absolutely and uniformly convergent throughout the half-plane $\sigma \geq M + \delta_1 > M$, is bounded in this half-plane. So that, the right side of (4) tends to $\varphi(0)$ as $s \to \infty$ in this half-plane. But by hypothesis $sf(s)$ also vanishes for a sequence of values of s in the same half-plane whose limit is infinity. Hence $\varphi(0) = 0$, that is

$$(5) \qquad sf(s) \equiv f_1(s) = \int_0^\infty \varphi'(x)\, e^{-sx}\, dx.$$

Now, applying the same argument to $f_1(s)$ as we did to $f(s)$ above, we can prove that $\varphi'(0) = 0$. Similarly, we can prove that $\varphi''(0) = \varphi'''(0) = \ldots = 0$. It follows, therefore, that the function $\varphi(x)$, analytic in the neighbourhood of the origin, vanishes along with all its successive derivatives at the origin, and so $\varphi(x)$ is identically zero. Consequently, $f(s)$ vanishes identically. This proves the theorem.

I wish to thank Dr. P. L. Srivastava, under whose guidance the paper was written.

ON THE SUMMABILITY OF FOURIER SERIES BY ARITHMETIC MEANS

By B. N. Prasad

Mathematics Department, Allahabad University

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1. Let $f(x)$ be a function periodic and integrable in the sense of Lebesgue in $(-\pi, \pi)$ and let its Fourier series be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad \ldots \quad \ldots \quad \ldots \quad (1\cdot 1)$$

In what follows we shall put

$$\phi(t) = \frac{f(x+2t)+f(x-2t)-2S}{2},$$

and

$$\Phi(t) = \int_0^t \phi(u)\, du.$$

The Cesàro summability, index unity, of a Fourier series at a point of continuity or simple discontinuity has been entirely disposed of by the classical theorem[1] of Fejér. Thereafter the question of finding criteria to be satisfied at points of discontinuity of the second kind became an important one. This question was taken up by Lebesgue who proved a more general theorem[2] which is applicable at all points at which

$$\lim_{t\to 0} \{f(x+2t)+f(x-2t)\}$$

exists and it may be valid even at points of discontinuity of the second

[1] Fejér, 4.

[2] Lebesgue, 9, 274.

kind. Later on Young generalized[1] Lebesgue's criterion; but curiously enough, until quite recently, this criterion of Young seems to have remained unnoticed.

All these criteria are essentially applicable *only* at points where the limit

$$\lim_{t \to o} \frac{\Phi(t)}{t} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1{\cdot}2)$$

exists. If we impose upon $f(x)$ the restriction of being bounded, or $f(x) > -A$, A being a constant, then, as shown by Hardy and Littlewood,[2] the existence of this limit is the necessary and sufficient condition for the Cesàro summability of every positive order.[3] Cases, however, are known in which the Fourier series is summable $(c, 1)$ even when the limit (1·2) does not exist.[4] But a general discussion of the summability $(c,1)$ of unrestricted functions $f(x)$ for which the limit (1·2) does not exist, seems to be lacking.

The object of this paper is to indicate a general simple method by which all these known criteria and several new ones can be obtained in a natural, connected manner. The underlying principle of the method is to reduce the discussion of the summability $(c,1)$ of the Fourier series corresponding to $\phi(t)$ to that of the ordinary convergence of the Fourier series corresponding to $\Phi(t)/t$ The method will be applicable even to cases in which the limit (1·2) does not exist.

2. We have for the partial sum of the Fourier series

$$s_n(x) = \frac{1}{2}a_0 + \sum_{m=1}^{m=n} (a_m \cos mx + b_m \sin mx)$$

[1] Young, **15**, 207, Corollary 4.

[2] Hardy *and* Littlewood, **7**.

[3] There seems to have been some misconception about the connection between the existence of the limit (1·2) and the summability $(c,1)$ of the corresponding Fourier series. For instance we find in Hobson's *Theory of Functions of a Real Variable*, Vol. 2 (1926), pp. 570-571, the statement that the condition $\int_o^t \phi(t)dt = o(t)$ "although *necessary*, is not sufficient, for the convergence $(c, 1)$ of the Fourier series" (italics ours). In fact the truth is that this condition is *neither necessary nor sufficient* for the summability $(c,1)$ of the Fourier series.

[4] For instance, see G. Prasad, **13**.

$$s_n(x) = \frac{1}{\pi}\int_0^{\pi/2}\left\{f(x+2t)+f(x-2t)\right\}\frac{\sin(2n+1)t}{\sin t}dt. \qquad (2\cdot1)$$

If $s_n(x)$ represents the Cesàro n^{th} partial sum of the Fourier series, we have

$$s_n(x)=\frac{s_0+s_1+s_2\ldots\ldots+s_{n-1}}{n}$$

$$=\frac{2}{n\pi}\int_0^{\pi/2}\frac{1}{2}\left\{f(x+2t)+f(x-2t)\right\}\left(\frac{\sin nt}{\sin t}\right)^2 dt.$$

Since
$$\int_0^{\pi/2}\left(\frac{\sin nt}{\sin t}\right)^2 dt=n\frac{\pi}{2},$$

$s_n(x)$ will tend to S as $n\longrightarrow\infty$, and the Fourier series will be summable $(c,1)$, if the integral

$$\frac{2}{n\pi}\int_0^{\pi/2}\left\{\frac{f(x+2t)+f(x-2t)-2S}{2}\right\}\left(\frac{\sin nt}{\sin t}\right)^2 dt.$$

$$=\frac{2}{n\pi}\int_0^{\frac{\pi}{2}}\phi(t)\left(\frac{\sin nt}{\sin t}\right)^2 dt. \qquad \ldots \quad \ldots \quad \ldots \quad (2\cdot2)$$

tends to zero, as $n\rightarrow\infty$.

3. Theorem I. *The Fourier series corresponding to* $\varphi(t)$ *will be summable* $(c, 1)$ *at a point* x, *if the Fourier series corresponding to* $\Phi(t)/t$ *is convergent in the ordinary sense at that point.*[1]

If δ be a number which may be chosen arbitrarily small, but greater than zero and independent of n, then we have

$$\frac{2}{n\pi}\int_\delta^{\pi/2}\phi(t)\left(\frac{\sin nt}{\sin t}\right)^2 dt = o(1).$$

[1] Long after I had proved this theorem I happened to see a theorem of W. H. Young which is substantially the same as this (see Young, 14, p, 266, theorem 8). But Young's proof and those of mine are of *entirely* different character from each other. Young first establishes a number of relations involving Fourier coefficients and therefrom he deduces his theorem. As will be observed, Young's proof is lengthy, complicated and very indirect, whereas the proofs given here are very concise, straightforward and (especially the second) very direct.

Hence the Fourier series will be summable $(c, 1)$ at x, if the integral

$$I_n = \frac{1}{n\pi}\int_0^\delta \phi(t)\left(\frac{\sin nt}{t}\right)^2 dt$$

is o (1). Integrating the above by parts, we get

$$I_n = \left[\frac{1}{n\pi}\Phi(t)\left(\frac{\sin nt}{t}\right)^2\right]_0^\delta - \frac{1}{n\pi}\int_0^\delta \Phi(t).\ \frac{d}{dt}\left(\frac{\sin nt}{t}\right)^2 dt$$

The term in the square brackets being o (1), the Fourier series will be summable $(c, 1)$, if each of the integrals

$$\frac{1}{\pi}\int_0^\delta \frac{\Phi(t)}{t}.\ \frac{\sin 2nt}{t}\, dt \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3{\cdot}1)$$

and

$$\frac{1}{n\pi}\int_0^\delta \frac{\Phi(t)}{t}\left(\frac{\sin nt}{t}\right)^2 dt \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3{\cdot}2)$$

is o (1).

Supposing $\Phi(t)/t$ to be integrable, the Fourier series corresponding to $\Phi(t)/t$ [in the same sense in which in the usual terminology, a Fourier series corresponds to $\phi(t)$] will be convergent, if (3·1) is o (1). In that case since (3·2) will behave as the Cesàro nth partial sum, (3·2) must also be o (1). Hence we conclude that whenever (3·1) is o (1), (3·2) is also o (1). This proves the theorem.

4. As we have just seen, the proof of the theorem consists in showing that whenever the integral (3·1) is o (1), the integral (3·2) is also o (1), and this we have proved by interpreting these integrals in terms of Fourier series. We can, however, prove this in a very direct manner which is free from the notions of Fourier series. We make the statement in a form connecting Dirichlet and Fejér integrals and as such it may prove of wider applicability.

Theorem II. *If the integral*

$$\int_0^\delta \chi(t)\frac{\sin 2nt}{t}\, dt$$

is o (1) *as* $n \to \infty$, *then also the integral*

$$\frac{1}{n}\int_0^\delta \chi(t)\left(\frac{\sin nt}{t}\right)^2 dt$$

is o (1).

The proof of it depends upon the following well-known simple lemma:—

Lemma. *If $c_n \to o$, as $n \to \infty$, then*

$$\sigma_n = \frac{c_1 + c_2 + c_3 + \ldots\ldots + c_n}{n}$$

also tends to zero, as $n \to \infty$.

Now taking $$c_n = \int_o^\delta \chi(t) \frac{\sin 2\,nt}{t}\, dt,$$

we have $$\sigma_n = \frac{1}{n}\int_o^\delta \frac{\chi(t)}{t} \cdot \frac{\sin nt.\ \sin(n+1)\,t}{\sin t}\, dt$$

$$= \frac{1}{n}\int_o^\delta \frac{\chi(t)}{t} \cdot \frac{\sin^2 nt}{\sin t} \cos t\, dt + \frac{1}{n}\int_o^\delta \frac{\chi(t)}{t} \sin nt \cos nt\, dt$$

$$= \frac{1}{n}\int_o^\delta \chi(t) \left(\frac{\sin nt}{t}\right)^2 dt + o(1) + \frac{1}{2n}\int_o^\delta \chi(t) \frac{\sin 2\,nt}{t}\, dt$$

$$= \frac{1}{n}\int_o^\delta \chi(t) \left(\frac{\sin nt}{t}\right)^2 dt + o(1) + o(1)$$

Thus $$\frac{1}{n}\int_o^\delta \chi(t) \left(\frac{\sin nt}{t}\right)^2 dt = o(1).$$

5. In order to deduce criteria for the summability $(c, 1)$ of Fourier series, let us divide the discussion according as (i) the $\lim_{t\to o} \Phi(t)/t$ is zero, or (ii) this limit does not exist, but $\Phi(t)/t$ possesses at $t = o$, a finite or infinite discontinuity of the second kind.

Taking the case (i), if we apply to

$$\frac{1}{\pi}\int_o^\delta \frac{\Phi(t)}{t} \cdot \frac{\sin 2\,nt}{t}\, dt$$

the various known standard tests of the ordinary convergence of a Fourier series, we shell get as many criteria for the summability $(c, 1)$. Out of these the application of Jordan's test[1] gives a criterion which is merely de la Vallée-Poussin's test[2] of the ordinary convergence and as such it must be rejected. We thus get the following criteria:—

[1] Jordan, 8.

[2] de la Vallée—Poussin, 1, also 2, 149.

By the application of Lipschitz's test[1] :—

(A) *The Fourier series will be summable* $(c, 1)$, *if*

$$\left| \int_0^t \phi(t)\, dt \right| = At^{k+1}$$

for all values of t not greater than some fixed positive number ϵ, A *and* k *being some fixed positive number.*

By the application of de la Vallée-Poussin's test :—

(B) *The Fourier series will be summable* $(c, 1)$, *if an interval* (o, ϵ) *can be found in which*

$$\frac{1}{t} \int_0^t dt \, \frac{1}{t} \int_0^t \phi(t)\, dt$$

is of bounded variation.

By the application of W H. Young's test[2] :—

(C) *The Fourier series will be summable* $(c, 1)$, *if*

$$\int_0^t \left| \phi(t) \right| dt = O(t).$$

This is the generalized form of Lebesgue's criterion due to Young A direct proof of it has been given by Pollard.[3]

By the application of Lebesgue's test[4] : —

(D) *The Fourier series will be summable* $(c, 1)$, *if*

$$\lim_{\epsilon \to o} \int_\epsilon^\xi \left| \frac{\Phi(t+\epsilon)}{t+\epsilon} - \frac{\Phi(t)}{t} \right| \frac{dt}{t} = o,$$

ξ *being a constant greater than* ϵ

From the consideration of the logical relations between the various criteria for the convergence of Fourier series,[5] it follows that (D) includes all the first three, (B) includes (A), while (B) and (C) are independent of each other.

6. Taking the case (ii) in which $\Phi(t)/t$ has got a discontinuity of the second kind at $t = o$, the problem is reduced to the discussion of the oscillating Dirichlet's integral

[1] Lipschitz, 10.
[2] Young, 15. 206.
[3] Pollard, 11
[4] Lebesgue, 9.
[5] Hardy, 5.

$$\int_{o}^{\delta} \frac{\Phi(t)}{t} \cdot \frac{\sin 2nt}{t}\, dt$$

By the application of Du Bois-Reymond's test,[1] we get the following criterion :—

(E) *If* $\qquad \dfrac{\Phi(t)}{t} = \rho(t) \cos \sigma(t),$

where each of $\rho(t)$ *and* $\sigma(t)$ *is monotone in* (o, δ) *and at least* $\sigma(t)$ *is unlimited, then the Fourier series will be summable* $(c, 1)$, *if*

$$\log \frac{1}{t} < \sigma(t) < \left(\frac{1}{t}\right)^{\triangle}$$

and $$\rho(t) < t \sqrt{\sigma''(t)}$$

where $\triangle$ *is any positive big number.*

By the application of the theorem of Riemann-Lebesgue, we get the following criterion[2] :—

(F) *The Fourier series will be summable* $(c, 1)$, *if the integral*

$$\int_{o}^{\delta} \left| \frac{\Phi(t)}{t^2} \right| dt$$

exists.

We give below an example in which the above criterion is applicable, although the $\lim\limits_{t \to o} \Phi(t)/t$ does not exist.

Example. Let

$$\Phi(t) = \frac{\sin^2 \left\{ n^4 \left(t - \frac{1}{n^2} \right) \pi \right\}}{n^2} \cdots \left(\frac{1}{n^2} \leq t \leq \frac{1}{n^2} + \frac{1}{n^4} \right),$$

and $\qquad \Phi(t) = o \qquad$ elsewhere.

Then $\phi(t) = \Phi'(t)$ is integrable, and so

$$\Phi(t) = \int_{o}^{t} \phi(u)\, du.$$

Also $$\int_{\frac{1}{n^2}}^{\frac{1}{n^2} + \frac{1}{n^4}} \left| \frac{\Phi(t)}{t^2} \right| dt < \frac{1}{n^2} \int_{\frac{1}{n^2}}^{\frac{1}{n^2} + \frac{1}{n^4}} \frac{dt}{t^2} < \frac{1}{n^2}$$

[1] Du Bois-Reymond, 3, 37 ; also see Hardy, 6, 259.

[2] Pollard has obtained a similar criterion for the Denjoy-Fourier series. See Pollard, 12.

so that the integral
$$\int_0^\delta \left| \frac{\Phi(t)}{t^2} \right| dt$$
exists; but $\Phi(t)/t$ does not tend to any definite limit as n tends to zero.

It may be mentioned that the criteria (F) and (C) are quite independent of each other, as may be easily verified by means of suitable examples. Thus for
$$\phi(t) = \tfrac{3}{2} t^{\frac{1}{2}} \sin \frac{1}{t} - \frac{1}{t^{\frac{1}{2}}} \cos \frac{1}{t}$$
the criterion (F) is satisfied but not (C). Again for
$$\phi(t) = \left(\log \frac{1}{t} \right)^{-1},$$
(C) holds, but not (F).

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NOTES ON BESSEL FUNCTIONS

By S. C. Mitra

Mathematics Department, Dacca University

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The object of the present note is to evaluate several infinite integrals involving Bessel Functions of order zero. It is believed that most of these results have not been published before.

1. We have the integrals[1]

$$\int_0^\infty y\, J_0\,(2xy)\, \cos\left(\frac{y^2}{k}\right) dy = \tfrac{1}{2}\, k \sin kx^2, \quad \ldots \quad \ldots \quad (1)$$

and

$$\int_0^\infty y\, J_0\,(2xy)\, \sin\left(\frac{y^2}{k}\right) dy = \tfrac{1}{2}\, k \cos kx.^2 \quad \ldots \quad \ldots \quad (2)$$

We shall take these two integrals as the starting point and deduce a number of interesting integrals.

In (2), let us divide both members by k and put $k = \cosh\,\theta$. Integrating* with respect to θ from zero to infinity, we get

$$\int_0^\infty \int_0^\infty y\, J_0\,(2xy)\, \sin\,(y^2 \operatorname{sech}\theta) \operatorname{sech}\,\theta\, d\theta\, dy$$

$$= \tfrac{1}{2} \int_0^\infty \cos\,(x^2 \cosh\,\theta)\, d\theta$$

$$= -\frac{\pi}{4}\; Y_0\,(x^2)$$

But

$$\int_0^\infty \sin\,(y^2 \operatorname{sech}\,\theta) \operatorname{sech}\theta\, d\,\theta$$

* The change in the order of integration can be easily justified.

becomes

$$\int_0^{\frac{\pi}{2}} \sin (y^2 \cos x)\ dx\ ,$$

by virtue of the substitution

$$\tan^{-1} (\tanh \tfrac{1}{2}\theta) = \tfrac{1}{2} x\ ,$$

and the latter integral is equal to $\frac{\pi}{2} H_0 (y^2)$, where $H_0(x)$ represents Struve's function of order zero.[2]

Therefore we have[3]

$$\int_0^{\infty} y\, J_0 (2xy)\, H_0 (y^2)\, dy = \tfrac{1}{2} Y_0 (x^2) \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

If we proceed with the first integral as before, we are able to deduce that[4]

$$\int_0^{\infty} y\, J_0 (2xy)\, J_0 (y^2)\, dy = \tfrac{1}{2} J_0 (x^2) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Again in (2), let us put $k=1$ and multiply both members by $\frac{x}{(x^4+4a^4)^{\frac{1}{2}}}$. Integrating both sides with respect to x from zero to infinity we get

$$\int_0^{\infty}\int_0^{\infty} \frac{xy\, J_0 (2xy) \sin y^2\, dxdy}{(x^4+4a^4)^{\frac{1}{2}}}$$

$$= \tfrac{1}{2} \int_0^{\infty} \frac{x \cos x^2\, dx}{(x^4+4a^4)^{\frac{1}{2}}} = \tfrac{1}{4} K_0 (2a^2)$$

But the left hand side can easily be shown to be equal to

$$\int_0^{\infty} y\, J_0 (2ay)\, K_0 (2ay) \sin\ y^2 dy.$$

We therefore arrive at the result[5]

$$\int_0^{\infty} y\, J_0 (2ay)\, K_0 (2ay) \sin y^2\, dy = \tfrac{1}{4} K_0 (2a^2) \qquad \ldots \quad (5)$$

In a similar manner we can prove the following results,

$$\int_0^{\infty} y\, J_0 (2ay)\, K_0 (2ay) \cos\ y^2\, dy = \frac{\pi}{8} \left[I_0 (2a^2) - L_0 (2a^2) \right]$$

where $L_0(x)$ bears the same relation to $H_0(x)$ as $I_0(x)$ bears to $J_0(x)$. ... ... (6)

$$\int_0^\infty y\, I_0(ay)\, K_0(ay) \sin y^2\, dy$$

$$= \frac{\pi}{8}\left[\sin \tfrac{1}{2}a^2\, J_0(\tfrac{1}{2}a^2) - \cos \tfrac{1}{2}a^2\, Y_0(\tfrac{1}{2}a^2)\right], \quad \ldots \quad \ldots \quad (7)$$

$$\int_0^\infty x\, J_0(2ax)\, K_0(2ax)\, H_0(x^2)\, dx = \frac{1}{4\pi} K_0{}^2(a^2), \quad \ldots \quad (8)$$

$$\int_0^\infty y\, K_0(2yk)\, J_0(y^2)\, dy = \frac{\pi}{8}\left[H_0(k^2) - Y_0(k^2)\right], \ldots \quad \ldots \quad (9)$$

$$\int_0^\infty y\, J_0(2ky)\, K_0(2ky)\, J_0(y^2)\, dy = \tfrac{1}{4} I_0(k^2)\, K_0(k^2), \quad \ldots \quad (10)$$

$$\int_0^\infty \left[I_0(2a^2 \cosh\theta) - L_0(2a^2 \cosh\theta)\right] d\theta = I_0(a^2) K_0(a^2). \quad \ldots \quad (11)$$

2. The Bessel-Fourier theorem states that with certain restrictions which are not important, a function $f(\rho)$ can be expressed by the double-integral

$$f(\rho) = \int_0^\infty\int_0^\infty xy\, J_n(x\rho)\, J_n(xy)\, f(y)\, dx dy \quad \text{where } n > -\tfrac{1}{2}.$$

Let $n=0$ and $f(y) = \dfrac{1}{(y^4+4k^4)^{\frac{1}{2}}}$.

We get

$$\int_0^\infty x\, J_0(x\rho)\, J_0(xk) K_0(xk)\, dx = \frac{1}{(\rho^4+4k^4)^{\frac{1}{2}}} \quad \ldots \quad \ldots \quad (11')$$

Now multiply both members of (10) by $k\, J_0(k\rho)$ and integrate with respect to k from zero to infinity.

We get

$$\int_0^\infty\int_0^\infty ky\, J_0(k\rho)\, J_0(y^2)\, J_0(2ky)\, K_0(2ky)\, dk\, dy$$

$$= \tfrac{1}{4}\int_0^\infty k\, J_0(k\rho)\, I_0(k^2)\, K_0(k^2)\, dk$$

But the double integral is seen to be equal to

$$\int_0^\infty \frac{y\, J_0\,(y^2)\, dy}{(\rho^4+64\, y^4)^{\frac{1}{2}}}$$

$$= \frac{1}{16}\, I_0\left(\frac{\rho^2}{16}\right) K_0\left(\frac{\rho^2}{16}\right).$$

Hence writing 4ρ for ρ, we have

$$\int_0^\infty k\, J_0\,(4k\rho)\, I_0\,(k^2)\, K_0\,(k^2)\, dk = \tfrac{1}{4}\, I_0\,(\rho^2)\, K_0\,(\rho^2) \qquad \ldots \quad (12)$$

Similarly we have

$$\int_0^\infty x\, J_0\,(2x\rho)\, [H_0\,(x^2) - Y_0\,(x^2)\,]\, dx = \tfrac{1}{2}\, [H_0\,(\rho^2) - Y_0\,(\rho^2)\,] \ldots \quad (13)$$

Whence taking into account the relation (3), we get

$$\int_0^\infty x\, J_0\,(2x\rho)\, Y_0\,(x^2)\, dx = -\tfrac{1}{2}\, H_0\,(\rho^2) \qquad \ldots \qquad \ldots \quad (14)$$

A similar procedure gives the result

$$\frac{\pi}{2}\int_0^\infty x\, J_0\,(x\rho)\, [I_0\,(\tfrac{1}{2}x^2) - L_0\,(\tfrac{1}{2}x^2)\,]\, dx = K_0\,(\tfrac{1}{2}\rho^2), \qquad \ldots \quad (15)$$

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ON THE ABSORPTION SPECTRUM OF HYDROGEN PEROXIDE VAPOUR

By R. S. Sharma

Physics Department, University of Allahabad

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Hydrogen peroxide is in many respects an anomalous chemical substance. It has generally strong oxidizing properties, but in several reactions, it acts as a reducing agent also. For example, the peroxide reduces ozone to oxygen, silver oxide to silver and lead dioxide to lead monoxide. This reducing property is, however, only apparent It is due to the fact, that the peroxide molecule readily parts with one oxygen atom which oxidises one oxygen atom in ozone, etc; the result being that both the reacting substances, *viz.*, the peroxide and ozone, etc., are reduced. This dual nature of hydrogen peroxide requires that both the oxygen atoms should not be symmetrically placed in the molecule. There are two alternative views about the constitution of the peroxide molecule.

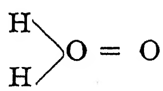
According to one view the constitution is

$$\mathrm{HO-OH}$$

or

$$\mathrm{HO{\equiv}OH}$$

While according to second view it is



For some time past workers in this laboratory have been studying the absorption spectra of various substances with a view to investigate the constitution and the nature of the force of binding in molecules. Very little work has, however, been done on the absorption spectrum of hydrogen peroxide vapour; although extensive work has been done with its solution in water. The work of Elder & Rideal[1] shows that the thermal decomposition of its vapour is mostly a surface action. Urey, Dawsey and Rice[2], however, have made an extensive study of the absorption spectra of its vapour and they find that there is a continuous absorption beginning at about 3000Å. They have further found that there is no trace of any band structure in the absorption spectrum. The fluorescence spectrum of its vapour, when it is illuminated with zinc spark

lines 2025Å to 2130Å, shows the well known water band at λ 3064 (0,0) which is associated with the OH molecule. Taking all this evidence into consideration they have come to the conclusion that the constitution of H_2O_2 is HO—OH, *i.e.* F_2 like, and the action of light on the molecule is as follows:—

$$HO-OH + h\nu = OH + OH\ (^2\Sigma)$$

In the present paper results of certain experiments on the action of light on the vapour of hydrogen peroxide are reported. The absorption spectrum has been studied in the ultra-violet region down to the limit of absorption of the atmosphere, while Urey and his co-workers confined their work up to λ2150. No attempt has, however, been made to measure the absorption coefficients. This has been very exhaustively done by Urey, Dawsey and Rice and it was thought that no improvement could be made on their work. An alternative suggestion for the photo-chemical dissociation of the peroxide molecule has, however, been put forward. This is based on the assumption that the structure of the peroxide molecule is $\begin{array}{l} H \\ \quad\rangle O = O \\ H \end{array}$

Two supplementary experiments have also been reported and they lend some support to the new point of view.

An attempt was made to study the recombination spectrum of hydrogen peroxide vapour. It may be thought that if the products of photo-dissociation recombined there will be an emission of energy corresponding to the heat of reformation. Thus we may expect a continuous spectrum beginning at a long wavelength limit and extending some distance towards the violet. This has been found to be the case.

Experimental

Three different samples of hydrogen peroxide were used in the study of the absorption spectra, *viz.*, 3%, 10% and 30% solutions supplied by Merck. The last is ordinarily known as perhydrol. The absorption chamber was a glass tube one centimeter in diameter having a length of 180 cms. The two ends of the tube were closed with quartz windows. The solution was contained in a bulb which was connected to the absorption chamber by a side tube. Another opening led to a pump which was kept running continuously. The continuous running of the pump served two purposes. Firstly it kept the pressure inside the absorption chamber low (about 2 or 3 mms.), thus removing water vapour and concentrating the peroxide solution progressively. The vapour pressure of the peroxide at the temperature of the absorption chamber, *i.e.*, at 20°C

is 1·5 mms. while that of water is 17·4 mms. The second advantage is that there is a fresh supply of vapour in the absorption chamber thus renewing the vapour which is reduced by the action of light.

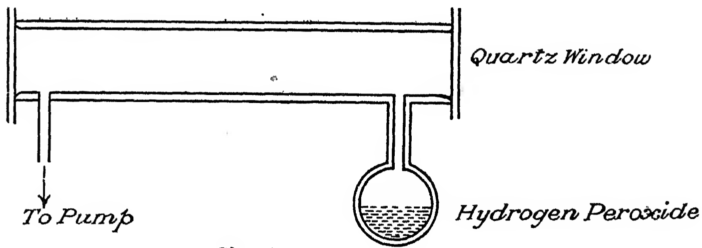


Fig. 1

For continuous light a Hydrogen tube run by a 2 K.W. transformer was used. Photographs were taken with a quartz E_3 spectrograph. Schumann plates were used for photographing the spectra. A copper arc was used as a standard.

The second part of the experiment consisted in studying the recombination spectrum of the peroxide vapour when it is illuminated with light of sufficiently short wavelength. A sketch of the apparatus used is given below.

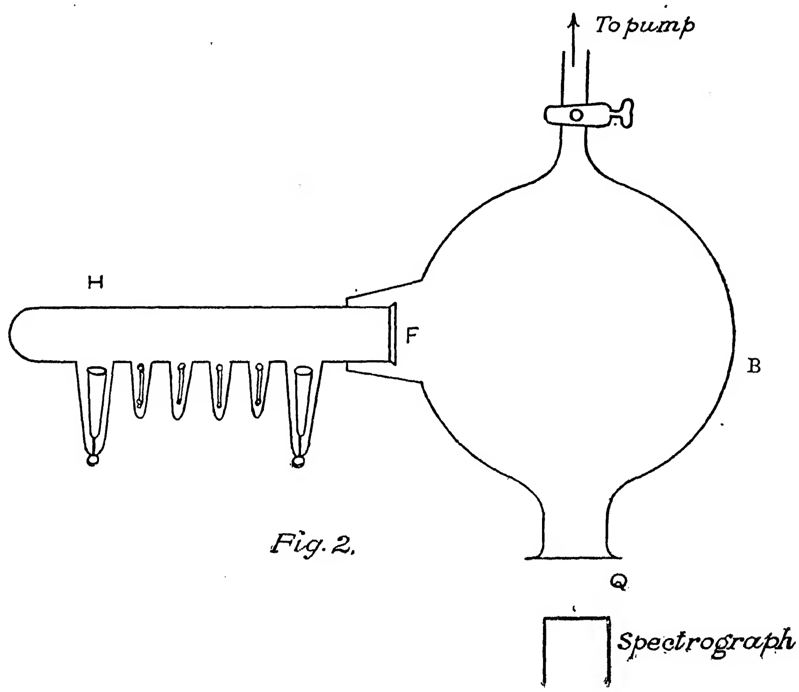


Fig. 2.

A Hydrogen tube H with a fluorite window F is sealed to a glass bulb B. At right angles to the length of the hydrogen tube is another opening which is closed with a quartz window Q. A side tube leads to a pump.

The inside of the bulb was thoroughly cleaned and was then rinsed with hydrogen peroxide. Next a small quantity of perhydrol was introduced in the bulb and the window was closed with the quartz plate Q. The bulb was then evacuated by a pump so that the pressure inside was a few mms. The stop-cock was then closed. The pump was run at intervals throughout the whole period of exposure. The bulb was covered on all sides to prevent any stray external light in the dark room from entering the bulb. The Hydrogen tube was run by a 2 K.W. transformer and exposure was taken through the window Q with a quartz spectrograph. The time of exposure was 40 hours. Another check exposure of the same duration, with the bulb empty, was also taken to test if the recombination spectra is really genuine and not due to internal reflection.

Results and Discussions

The results of this investigation are as follows:—

1. The absorption spectrum shows a sharp cut at 2055 Å.

This corresponds 139 kcal.

2. The recombination spectrum shows a continuous patch of light beginning at about 4800Å and extending to about 3900Å.

I shall now try to interpret these results in the light of Urey's model. We suppose that the constitution of the peroxide molecule is HO−OH or F_2 like. But the absorption spectrum of F_2 consists of bands followed by continuous absorption from a long wavelength limit which marks its splitting into F $(^2P_{\frac{3}{2}})$ + F $(^2P_{\frac{1}{2}})$. In the case of H_2O_2 we get no bands in absorption. Hence the Franck-Condon diagrams should be as in figure 3.

The normal state of the molecule is represented by the curve L. By the absorption of light it passes from the point B in the curve L to the point C in the curve M which represents the next higher state of the molecule. This curve has no minimum. The molecule will therefore dissociate into OH $(^2\Pi_{\frac{3}{2}})$ + OH $(^2\Pi_{\frac{1}{2}})$. The molecule may, however, also

pass to the point D in the second higher curve N and may dissociate into OH ($^2\Pi_{\frac{3}{2}}$) + OH ($^2\Sigma$). The energy of excitation of the λ 3064 (0, 0) band

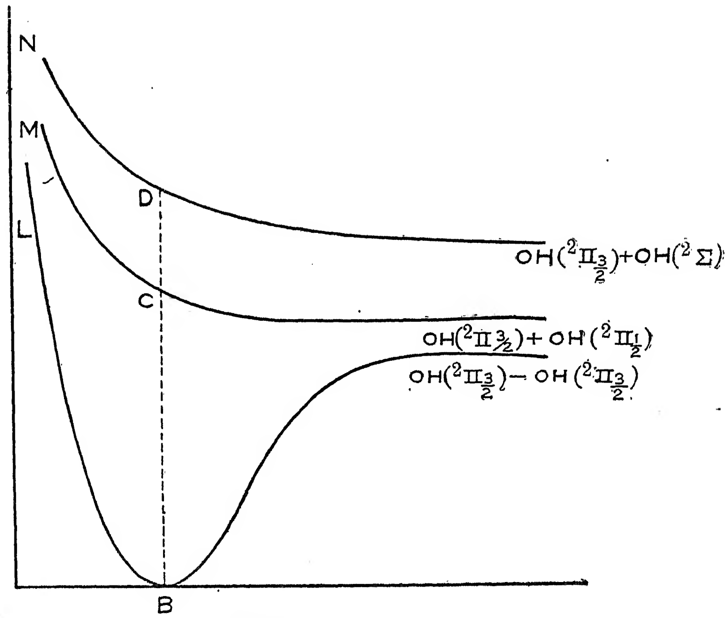


Fig. 3.

which is due to the excitation of OH from ($^2\Pi_{\frac{3}{2}}$) to $^2\Sigma$ is 93·4 kcal. Hence supposing that the cut observed by me at λ2055 corresponds to the dissociation in the state D the heat of dissociation of H_2O_2 into two normal OH molecules is 139·1 − 93·4 = 45·7 kcal.

But it can be shown that this does not agree with the value of the heat of dissociation obtained from other data. This can be approximately found as follows :—

$$\begin{array}{llll}
(a) & \mathrm{HOOH} & = \mathrm{HOH} + \tfrac{1}{2}\mathrm{O}_2 & + 25{\cdot}2 \text{ kcal.} \\
(b) & 2\mathrm{HOH} & = 2\mathrm{H} \quad + 2\mathrm{OH} & - 222 \;\;\text{,,} \\
(c) & 2\mathrm{H} & = \mathrm{H}_2 & + 101 \;\;\text{,,} \\
(d) & \mathrm{H}_2 + \tfrac{1}{2}\mathrm{O}_2 & = \mathrm{H}_2\mathrm{O} & + 57 \;\;\text{,,} \\
\hline
(e) & \mathrm{HOOH} & = 2\mathrm{OH} & - 38{\cdot}8 \text{ kcal.}
\end{array}$$

The thermo-chemical figures involved in (*a*) have been taken from the tables of Landolt and Börnstein. Those in (*b*) have been given by

Bonhoeffer and Reicharet[3] and Bonhoeffer and Haber[4]. The heat of formation of H_2O given in (*d*) has been taken from Frankenburger and Klinkhardt.[5]

The disagreement between this value, *i.e.*, 38 kcal and that obtained from absorption experiments, *viz.*, 45·6 kcal is appreciable.

Passing on now to the other constitutional formula for the peroxide molecule, *i.e.* $\begin{matrix} H \\ H \end{matrix}\!\!>O=O$ the possible photo-chemical reactions are as follows:—

$$(f) \quad \begin{matrix} H \\ H \end{matrix}\!\!>O=O+h\nu = H + H-O=O$$

$$(g) \quad HOOH \quad +h\nu' = 2H+O_2$$

The energy corresponding to the reaction (*f*) cannot be calculated, but that involved in (*g*) can be easily found as follows:—

$$\begin{array}{llll} (h) & HOOH = H_2+O_2 & & -\ 34 \text{ kcal} \\ (c) & H_2 & =2H & -102 \text{ kcal} \\ \hline (i) & HOOH = 2H & & -136 \text{ kcal.} \end{array}$$

The thermo-chemical data in (*h*) is taken from Urey's[2] paper. The heat of dissociation of H_2 is taken as 102 kcal[6].

The energy value 136 kcal. that we get from (*i*) almost correspond to the continuous absorption at 2055Å obtained in the present investigation.

On this view there should be a second absorption corresponding to the photo-chemical reaction in (*f*). Now Urey and his co-workers have taken the beginning of absorption to be at 3000Å from the measurement of absorption coefficients. They have, however, remarked that the absorption by vapour might begin much earlier (towards the red side), almost at the same place as the absorption by the aqueous solution of H_2O_2, *i.e.*, at 3750Å. The recombination spectrum in the present investigation which begins at 4800Å can be interpreted as being due to the emission of energy when the products of photo-chemical reaction (*f*) recombined to form the H_2O_2 molecule. The energy value corresponding to 4800Å is 59·6 kcal. This is then the energy required to remove the first hydrogen atom from the peroxide molecule. This is not an unlikely value, for in the case of H_2O molecule we know that the

removal of the first hydrogen atom requires a little less energy than half the energy required to remove both the hydrogen atoms. This is shown below[4]:—

$$H_2O = H + OH - 111 \text{ kcal}$$
$$HO = H + O \quad -123 \quad ,,$$

To test the presence of atomic hydrogen (if our point of view of the photo-chemical dissociation of the peroxide vapour be correct) a small piece of cupric oxide was placed in the absorption chamber. The vapour was then illuminated with carbon arc light which gives the strong carbon line $\lambda 1931$. The cupric oxide acquired a slight reddish tinge. This may be due to the reduction of the cupric to the cuprous oxide.

Besides if the constitution of H_2O_2 is HO-OH, *i.e.*, F_2 like, it should have no electric moment as the electric moment of the halogen molecules is zero. Linton and Maas[7] have recently determined the electric moment of H_2O_2. They find it to be $2{\cdot}13 \times 10^{-18}$ units at 25°C. The electric moment[8] of H_2O, as they find it, is $1{\cdot}90 \times 10^{-18}$ units and is of the same order of magnitude as that of H_2O_2. These authors are therefore inclined to the view that the constitution of H_2O_2 is $H_2O{=}O$. This agrees with our conclusion.

Summary

1. The absorption spectrum of hydrogen peroxide vapour shows a sharp continuous absorption beginning at 2055Å. This corresponds to 139·1 kcal.

2. The recombination spectrum shows a continuous emission of light beginning at 4800Å. This corresponds to 59·6 kcal.

3. The sharp cut at 2055Å is interpreted as due to the following photo-chemical dissociation :—

$$H_2O_2 + h\nu = 2H + O_2$$
$$\text{where } h\nu = 136{\cdot}1 \text{ kcal.}$$

4. The recombination spectrum is interpreted as to be due to the energy emitted in the process:—

$$H + HOO = H_2 O_2 + h\nu \ (59{\cdot}6 \text{ kcal}).$$

5. The slight reduction of cupric oxide lends some support to the view that atomic hydrogen is formed in the photo-chemical dissociation of the peroxide vapour.

My best thanks are due to Prof. M. N. Saha, F.R.S., for his kind interest and guidance in this work.

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THE APPLICATION OF FRANCK CONDON PRINCIPLE TO CONTINUOUS ABSORPTION SPECTRA OF DIATOMIC MOLECULES

By Hrishikesha Trivedi

Physics Department, Allahabad University.

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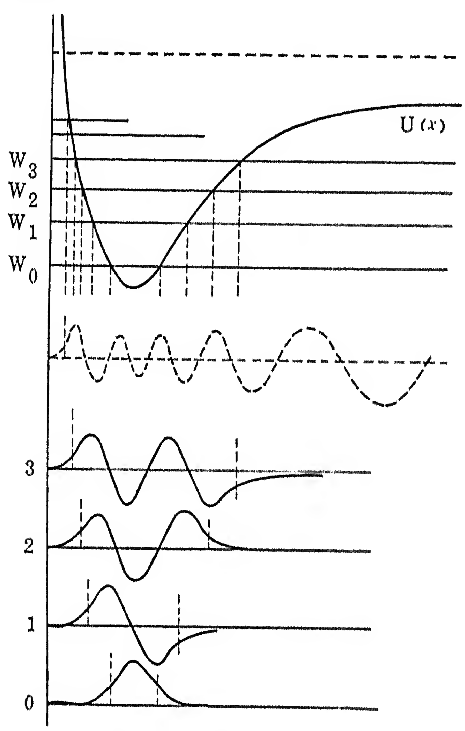
Much experimental work has been done recently on the continuous absorption spectra of various molecular compounds. Franck and his co-workers[1] mostly confined their attention to the point of maximum absorption. Attention to the importance of the beginning of absorption was first drawn by workers in this laboratory. These observations were applied to the determination of the energies of dissociation of the molecules from the value of the initial absorption frequency. The process of optical dissociation, which gives rise to such a spectrum, is supposed to be governed by Franck's principle. According to this principle, in the act of absorption of light quanta, the electron passes to the higher level so quickly that the vibration and rotation terms are not affected at all. Condon[2] has attempted to give a picture of this principle. According to him the process of optical dissociation can best be understood by considering curves showing the variation of the potential energy of the components with the distance between them (internuclear distance). At all points of this potential energy curve the molecule will have the same electronic energy. The state of vibration of the molecule is represented by drawing horizontal lines the heights of which above the abscissa axis represent the energy of vibration of various states. The vibration ψ function for any stable di-atomic molecule in a particular electronic state was found by Condon[2] and Hutchisson[3] to have a maximum value at those points where the above mentioned horizontal lines cross the potential energy curve (Fig. 1). In accordance with Franck's hypothesis, that the distance is not changed in the act of absorption, Condon represented the transitions from one electronic state of the molecule to its another electronic state brought about by the absorption of light by vertical arrows. These

arrows originated from points of maximum value of the vibration ψ function of the lower electronic state. This application of Franck's principle by Condon was found to be very much successful in the case of band spectra of di-atomic molecules. According to Condon, those transitions should be most intense for which the maxima of ψ functions in the two states coincided. Experimental results confirmed this, although there are exceptions.



Eigen-values and Eigen-functions of a typical system with one degree of freedom (Smekal's Quanten theorie, S. 570).
Fig. 1

The variation of intensity in the continuous absorption spectrum of a di-atomic molecule was also explained by Condon on these very lines. The molecules considered were the halogens. Condon tests the validity of his method as applied to continuous spectra by examining if his method gave the right order of magnitude for the breadth of the continuous band indicating the dissociation of a molecule. According to him the order of magnitude involved is approximately equal to the interval on the frequency scale given by "reflecting" the initial state vibration wave-function in the potential energy curve of the final state. Condon considers the zero state to be non-vibrating and therefore assumes that its wave-function is represented by a Gauss error curve.*

*It is proved in Wave-mechanics that the wave-function of a linear oscillator (vibration quantum number n) is given by

$$\psi_n = e^{-\frac{x^2}{2}} \cdot \mathrm{H}_n(x)$$

$$\text{where } x = r\sqrt[4]{b} \quad ; \quad b = \frac{16\pi^4\nu_0{}^2}{h^2}$$

and $\mathrm{H}_n(x)$ are the Hermite functions defined as

$$\mathrm{H}_n(x) = (-1)^n \, e^{x^2} \frac{d^n}{dx^n} \, e^{-x^2}$$

$$\text{for } n=1 \quad , \quad \mathrm{H}_n(x) = 1$$

Hence ψ_0 = Gaussian error curve.

The actual procedure adopted by Condon in the operation which he calls "reflection" seems to be one given by figure 2. Vertical lines are drawn from the various points A, B, C, D, E on the Gauss error curve 1, so as to touch the upper potential energy curve, wherefrom horizontal

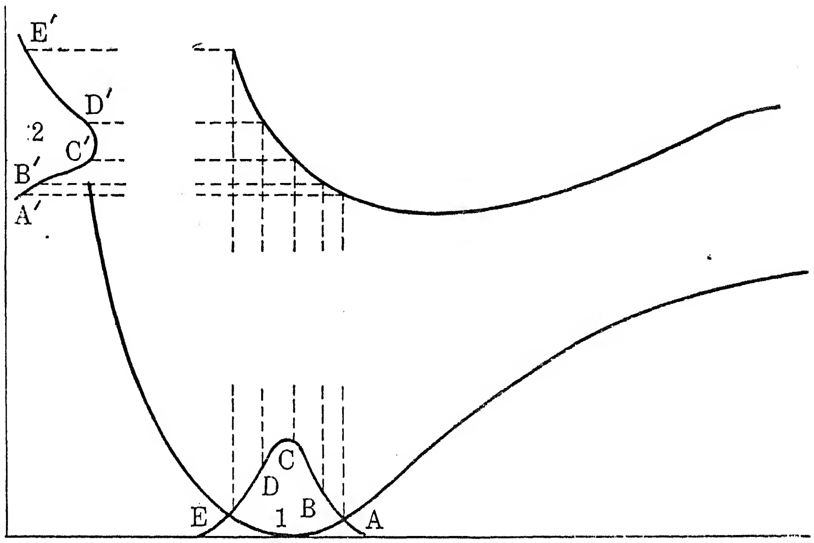


Fig. 2

lines are drawn so as to cross the axis of Y. On these lines are marked points whose distances from the Y axis is just the same as are those of the corresponding points A, B, C, D, E on the Gauss error curve 1 from the X axis. This curve 2 is roughly similar to the curve 1, and as Condon takes A′ E′ to be the width of the continuous absorption band in the wave number scale marked along the Y axis, the curve 2 represents the transitional probabilities from the lower state to the upper one.

The assumption of Condon that the zero state is non-vibrating does not, however, always correspond to the experimental conditions. The temperatures at which the experiments have been conducted to determine the intensity of continuous absorption band are such that a considerable fraction of molecules may pass to higher vibrational states. For such states, the wave-function will not be a Gauss error curve but one with at least two maxima ($n=1$). Taking up the same line of argument as Condon did in the case where the vibration wave function was a Gauss error curve (zero vibration level) it can be easily seen that the transition probabilities from the lower to the upper state will be given by a curve having two maxima, when the molecules in the lower state are in the

first vibration level and correspondingly when the vibration wave-function of the lower state is not a Gauss error but a curve with two maxima (*cf.* Fig. 1). The absorption spectrum will, therefore, have two maxima separated by a region where the absorption is smaller than on either side of it. Such a state of affairs has not been traced.

It may be remarked at this stage that there are certain types of continuous absorption spectra which have a fluctuation in their intensities at the long wavelength beginning. They have been classified by Finckelnburg[4] under class II. There has been observed yet another kind of continuous absorption spectrum with fluctuations of intensity at the long wavelength beginning of it by Sommermeyer[5] in the case of some alkali halides. The fluctuations are very gradual and wide. It can, however, be seen very easily that the intensities of both these types of absorption spectra cannot be obtained when the vibration wave-function of the lower state possessing several maxima (vibration quantum number high) (*cf.* Fig. 1) is reflected in the upper wave-function. The actual process of reflection shows that if the intensity fluctuation is not to be wider than what is observed, the upper potential curve is to be flat and the fluctuation will extend all over the length of the continuous absorption spectrum which itself will not be very wide. Experiments show us, however, that the intensity fluctuation is only at the long wavelength beginning of the absorption spectra, which in itself is sufficiently wide in the case of Finckelnburg's class II spectra, and passes outside the region investigated in the case of alkali halides. We can thus see that such fluctuations in the intensity of an absorption spectrum cannot be explained by the application of Condon's method.

This disagreement between theory and experiment may be explained in the following way. The left side of the upper curve may be so steep that the ultra-violet end of the spectrum may fall outside the region which is available for investigation, so that Condon's process will give us an absorption spectrum which actually corresponds to facts.

In accepting this explanation, however, we will have to consider one thing. The beginning of the absorption corresponds tolerably well to the heat of dissociation of practically all the di-atomic molecules so far investigated as determined thermochemically or by any other reliable means. This fact requires the upper curve to be very nearly horizontal from the point P vertically above the beginning of the first maximum of the vibration wave-function of the lower curve (Fig. 3) In order to make it compatible with the above-needed steepness of the upper curve we have to make either of the following assumptions. First, the upper curve

suddenly becomes steep to the left of the point P. Secondly, if the curve does not become steep suddenly to the left of P we take Condon's principle to be invalid; for it was only for keeping it valid and still to explain the extension of the continuous spectrum to infinity that we had to suppose that the curve suddenly became steep to the left of P. Thirdly, we can suppose the upper curve to have very nearly the same slope throughout and disregard the assumptions according to which the curve beyond P is horizontal. If we assume the third alternative the energy which corresponds to the beginning of absorption cannot give the heat of dissociation of the molecule.

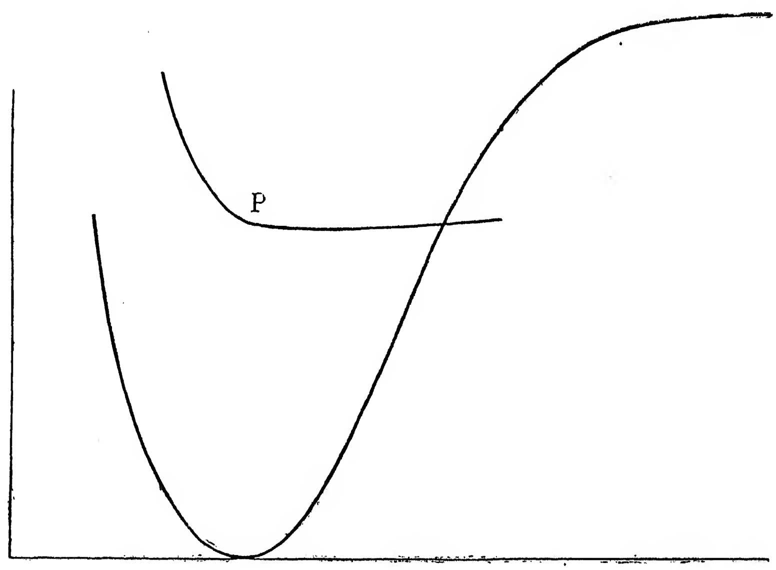


Fig. 3

In the state of the molecule which is represented by the upper curve we have as yet no knowledge of any molecular force which will make the slope of the curve to be changed by such a degree and such a suddenness at a particular point P. It would be much safe to assume that there is no such transition to the steep slope of the curve. If we drop the explanation for which the steepness had to be introduced we are forced to the possibility that Condon's procedure is invalid for many cases of continuous absorption spectra of di-atomic molecules. If, however, we reject the explanation which necessitates the upper curve to be horizontal beyond P we let the very reliable and very plausible work of Franck[1] and his school fall flat to the ground. The results obtained by Franck

and his co-workers in interpreting the continuous absorption in terms of the energy of dissociation of the molecule are so many and of such a quantitative precision as to merit their parent theory being taken to be true.

The aim of the present paper is to land us out of this quagmire. To do this the following procedure will be adopted. There are various kinds of continuous spectra observed in molecules. Finckelnburg[4] has classified them in six different classes It is the purpose of this communication to deal with homogeneous continuum which remains purely continuous even if the temperature and pressure of the absorbing gas varies. This has been classified by Finckelnburg as class I. As this type of spectra is obtained mostly in absorption the obvious conclusion is that the molecules are extremely unstable in the upper state, and the binding forces will be only repulsive ones.

Such a repulsive potential has been computed for atomic hydrogen,[6] helium,[7] and for molecules.[8] It appears possible to represent it in the range of importance in the mutual repulsion of the atoms by the approximate formula

$$A + De^{-2ar}$$

where A, D and a are constants, r the nuclear distance and e the base of logarithm.

The next step to be taken is the calculation of the wave-function for the unstable state of the molecule. The wave equation of the molecule is as follows:

$$\Delta \psi + \frac{8\pi^2\mu}{h^2}(E-V)\psi = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where E = energy constant, V = potential energy expressed as a function of the coordinates.

Considering it to be a problem of boundary values with spherical symmetry and introducing polar coordinates r, θ, ϕ, we obtain from equation (1)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$+\frac{8\pi^2\mu}{h^2}\left(E - A - De^{-2ar}\right)\psi = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where μ = reduced mass.

Integrating this equation by the method of separation of variables, we set

$$\psi = R(r)\, \theta(\theta)\, \varphi\, (\phi).$$

Considering first $\varphi\ (\phi)$ we see that, as it does not occur explicitly in the equation (2)

$$\varphi\ (\phi) = e^{\pm im\phi}.$$

The postulate of uniformity leads to the integral values of m.

Substituting this value of φ in (2) and considering $\theta\,(\theta)$ we get

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial\theta}{\partial\theta}\right\}+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\theta = 0$$

where λ is a constant $= n(n+1)$; n being a positive integer. We get therefrom

$$\theta\ (\theta) = \mathbf{P}_n^m\ (\cos\theta)$$

Now let the term containing r in the equation (2)

be
$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\ (r\psi).$$

We now set
$$r\,\psi = F.\ \mathbf{P}_n^m\ (\cos\theta).\ e^{\pm im\,\phi}$$

As the final state is continuous, the m in the above expression, denoting the rotational quantum number, can be placed equal to zero, therefore

$$r\,\psi = F.\ \mathbf{P}_n^o(\cos\theta)$$

whence we get the wave-equation

$$\frac{d^2F}{d\,r^2}+\frac{8\pi^2\mu}{h^2}\left(W-De^{-2ar}\right)F = 0 \qquad \dots \quad \dots \quad (3)$$

where $W = E - A$

Making a transformation $y = e^{-ar}$, we have

$$\frac{dF}{dr} = \frac{dF}{dy}.\frac{dy}{dr} = \frac{dF}{dy}.\left(-\,a\,e^{-ar}\right)$$

$$\frac{d^2F}{dr^2} = \frac{d^2F}{dy^2}.\frac{dy}{dr}.\left(-\,ae^{-ar}\right) + a^2e^{-ar}.\frac{d\,F}{dy}$$

whence equation (3) is transformed to

$$a^2y^2\frac{d^2F}{dy^2} + a^2y\frac{dF}{dy} + \frac{8\pi^2\mu}{h^2}(W - Dy^2)\,F = 0 \qquad \ldots \qquad \ldots \qquad (4)$$

or

$$\frac{d^2F}{dy^2} + \frac{1}{y}\frac{dF}{dy} + \frac{8\pi^2\mu}{a^2h^2}\left(\frac{W}{y^2} - D\right)F = 0 \ \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (5)$$

According to the meaning of r the boundary points of the region in y are the values $y=0$ and $y=\infty$. We can easily see that $y=0$ is a pole.

Schrödinger's[9] criterion for F as determined by equation (5) is that it will be finite and single-valued throughout the space. For the problem of finding the solution which remains finite everywhere, it is necessary to see that the solution is finite at the points of singularity for the differential equation.

The equation (5) takes the form

$$y^2\frac{d^2F}{dy^2} + y\frac{dF}{dy} + \frac{8\pi^2\mu}{a^2h^2}(W - Dy^2)\,F = 0 \qquad \ldots \qquad \ldots \qquad (6)$$

Putting

$$\frac{8\pi^2\mu D}{a^2h^2} = -\lambda^2;\ \lambda y = x;$$

and

$$\frac{8\pi^2\mu W}{a^2h^2} = -\nu^2$$

we get the equation (6) as

$$x^2\frac{d^2F}{dx^2} + x\frac{dF}{dx} + (x^2 - \nu^2)\,x = 0 \ \ldots \qquad \ldots \qquad \ldots \qquad (7)$$

Let us now construct a solution of (7) which is valid near $y=x=0$; the form assumed for such a solution is a series of ascending powers of x, say

$$F = \sum_{m=o}^{\infty} C_m\, x^{a+m}$$

where the index α and the coefficients C_m are to be determined, with the proviso that C_0 is not zero.

For brevity the differential operator which occurs in (7) will be called ∇v, so that

$$\nabla v \equiv z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + z^2 - v^2.$$

It can be seen easily that

$$\nabla v \sum_{m=o}^{\infty} C_m z^{\alpha+m} = \sum_{m=o}^{\infty} C_m \{ (\alpha+m)^2 - v^2 \} z^{\alpha+m} + \sum_{m=o}^{\infty} C_m z^{\alpha+m+2}$$

The expression on the right reduces to the first term of the first series, namely, $C_0 (\alpha^2 - v^2) z^{\alpha}$, if we choose the coefficients C_m so that the coefficients of corresponding powers of z in the two series on the right cancel. This choice gives the system of equations

$$\left.\begin{array}{ll} C_1 \{ (\alpha+1)^2 - v^2 \} & = 0 \\ C_2 \{ (\alpha+2)^2 - v^2 \} + C_0 & = 0 \\ C_3 \{ (\alpha+3)^2 - v^2 \} + C_1 & = 0 \\ \dots\dots\dots\dots\dots\dots & \\ C_m \{ (\alpha+m)^2 - v^2 \} + C_{m-2} & = 0 \\ \dots\dots\dots\dots\dots\dots & \end{array}\right\} \dots \quad \dots \quad \dots \quad (8)$$

From this result it is evident that the postulated series can be a solution of (7) only if $\alpha = \pm v$; for C_0 is not zero, and z^{α} vanishes only for exceptional values of z. We must further take $\alpha = +v$ in order that F may be finite for $z=0$, when v is a positive quantity.

It can now be very easily seen that

$$C_0 z^{\nu} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m (\frac{1}{2} z)^{2m}}{m!\,(v+1)(v+2)(v+3)\dots\dots(v+m)} \right] \quad .. \quad (9)$$

is a formal solution of (7).

Any value independent of z may be assigned to the constant C_0. Putting it equal to[10]

$$\frac{1}{2^{\nu}\ \Gamma(v+1)}$$

we get the series (9) in the form

$$\sum_{m=o}^{\infty} \frac{(-1)^m (\frac{1}{2} z)^{\nu+2m}}{m!\ \Gamma(^1+m+1)} = \mathbf{J}_{\nu}(z).$$

N.B.—All the ν's used above denote the same quantity, although at some places they appear different from the rest.

This series is a solution of the equation (7) which is valid near $y-x=0$ (the pole of the equation) for all values of x and for all positive values of ν.

The series which defines $\mathbf{J}_\nu(x)$ converges absolutely and uniformly in any closed domain of values of x, and in any bounded domain of values of ν.

When $|\nu| \leq N$ and $|x| \leq \Delta$, the test ratio for this series is

$$\left|\frac{-\frac{1}{4}x^2}{m(\nu+m)}\right| \leq \frac{\frac{1}{4}\Delta^2}{m(m-N)} < 1$$

whenever m is taken to be greater than the positive root of the equation

$$m^2 - mN - \tfrac{1}{4}\Delta^2 = 0.$$

This choice of m being independent of ν and x, the result stated follows from the test of Weirstrass Hence $\mathbf{J}_\nu(x)$ is an analytic function of x for all values of x and it is an analytic function of ν for all values of ν. Whence we can see that $\mathbf{J}_\nu(x)$ remains a finite quantity and is a solution of the equation (7) for all values of ν and ν changes its value continuously.

In the famous equations solved by Schrödinger each one of them was solvable only when a particular parameter involving the total energy had certain discrete values. This restriction of the total energy having only certain discrete values was interpreted by him in terms of line or discrete band spectra. In the case under consideration the parameter ν involving the total energy can have all values and always the wave-equation can be solved, therefore reasoning in an analogous way we can say that it represents continuous spectrum.

This continuous spectrum begins where the value of ν is equal to zero and extends onwards with its various positive values. When $\nu=0$, W must also be equal to zero, *i.e.*, $E-A$ must be equal to zero. The energy corresponding to A, being the heat of the dissociation of the molecule in that state, will in the Condon diagram be represented by the horizontal asymptote to the potential energy curve. The vertical height of this asymptote above the minimum position of the lowest state represents the heat of dissociation of the molecule in that excited state. The above consideration (*i.e.*, the continuous spectrum begins at $W=0$) shows that the continuous spectrum begins at this horizontal asymptote representing the heat of dissociation of the molecule in that excited state, no matter how much below it lies from the actual curve giving the relation between the potential energy of the molecule and its inter-nuclear distance, if we stick to the Condon diagram.

This fact provides us with the justification of Datta's[11] extrapolation of the curve relating the absorption coefficient with the wavelengths of the absorbed light which had previously appeared more or less *ad hoc.* He reached, thereby, to a beginning of absorption which corresponded to the heat of dissociation of the molecule. For we can easily see that there is no sense in the extrapolation unless there is a definite value of the vibration ψ function for the upper state different from zero for energies down to the heat of dissociation of the molecule.

Now we come to the representation of the process of the production of this continuous spectrum. Condon assumed that the transition took place only between two potential energy curves. There was to be no transition between any point lying on the lower potential energy curve and some other which did not lie on the upper potential energy curve. We have, however, seen above that the transition can be to a point which is lower than the nearest vertical on the upper curve. The extension of the continuous spectrum beyond that point to any value shows that points lying on the other side of the upper curve have also a transition probability to the lower curve. We are not justified, therefore, in forcing the points having a transition probability to the lower state to lie on a curve. They, on the contrary, seem to lie on a surface. The continuous spectrum which we observe is due to a transition from points on the lower curve lying on both sides of its minimum, where the vibration ψ function has maximum values, but we are not in a position to distinguish the portion due to each one of them. In fact it would be better not to regard this question in terms of potential energy curves but in terms of Schrödinger's equations for the various states of the molecule and their respective characteristic functions.

In order, therefore, to explain the continuous absorption by di-atomic molecules we have not to make any more assumptions excepting the following. First, the upper repulsive curve is given by a formula of the type $A+De^{-2ar}$, an assumption which we have seen above to be justified. Secondly, the energy given by A is the energy of dissociation of the molecule in that state. All other postulates and assumptions seem unnecessary.

We shall deal with the various transition probabilities and the intensities to be expected in another communication.

ACKNOWLEDGEMENTS

It is the pleasant duty of the author to acknowledge his indebtedness to Prof. M. N. Saha, F.R.S., for his kindness on him and his interest in this work.

SUMMARY

The homogeneous absorption spectrum of di-atomic molecules has been considered by Condon. Certain difficulties appear in that treatment. They have been pointed out. The problem is re-considered from the point of view of wave-mechanics. The potential energy of the molecule in the upper unstable electronic state is taken to be given by $A+De^{-2ar}$. A wave equation of the molecule for this state is formed and its solution shows that the ψ function has some value, different from zero, for all values of energy $(E-A)$ greater than zero, where E=energy constant. It is shown that the beginning of absorption gives the heat of dissociation of the molecule and the absorption will extend unrestricted to infinity.

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THE QUANTUM ANALOGUE OF A THEOREM OF POISSON IN CLASSICAL DYNAMICS

By D. S. Kothari

Physics Department, University of Allahabad

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The discovery by Dirac of the quantum analogue of Poisson-bracket enables one to readily transform theorems in classical dynamics to the corresponding theorems in Quantum Mechanics. As one more (rather simple but elegant) illustration of this procedure we consider the following "classical" theorem of Poisson [1]. If ϕ and Ψ are two integrals of a dynamical system, then the Poisson-bracket $[\phi, \Psi]$ is constant throughout the motion, *i.e*, the equation $[\phi, \Psi] = \text{constant}$ constitutes a new integral of the system. The proof of this important theorem in classical dynamics need not be reproduced here. The Quantum-mechanical proof is very simple.

Let α and β be two observables that are constants of the motion, *i.e.*, α and β commute with the Hamiltonian H: but α and β do not necessarily commute with each other.

Hence we have,

$$\alpha H - H\alpha = 0$$
$$\beta H - H\beta = 0$$

and therefore

$$\begin{aligned} &(\alpha\beta - \beta\alpha)\, H - H\, (\alpha\beta - \beta\alpha) \\ &= \alpha\, (\beta H - H\beta) + \beta\, (\alpha H - H\alpha) \\ &= 0 \end{aligned}$$

i.e, the expression $(\alpha\beta - \beta\alpha)$ is also a constant of the motion. But $(\alpha\beta - \beta\alpha)$ is the quantum definition of the Poisson-bracket $[\alpha, \beta]$ and hence the above theorem is proved.

Let us consider an example. Suppose the components of angular momentum M_x and M_y are constants of the motion,

$$i.e., \qquad M_x H - H M_x = 0$$

$$M_y H - H M_y = 0.$$

Then, it follows from the theorem that $[M_x, M_y]$ is also a constant of the motion. But from the well-known relation

$$M \times M = i h M$$

where M is the angular momentum vector, it follows that $[M_x, M_y] = ih M_z$ and hence M_z is also a constant of the motion.

This result is what one would expect from classical analogy.

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CHEMICAL EXAMINATION OF *PUNAR-NAVA* OR *BOERHAAVIA DIFFUSA, LINN.*

BY

RADHA RAMAN AGARWAL AND SIKHIBHUSHAN DUTT.

Chemistry Department, Allahabad University.

Received March 13, 1934.

Boerhaavia diffusa known as Punar-nava in Sanskrit and Bengali, and Sant, Thikri or Gadhva-purna in Hindusthani, is a plant of the natural order Nyctaginæ. It is grown throughout India from Punjab to Assam and south to Travancore. It is diffusely branched herb, with rather thick leaves and white or red flowers; and grows profusely on barren lands, being a creeping weed.

As regards its medicinal properties Punar-nava has been mentioned by *Sushruta* and other older Sanskrit writers and hence is a plant of long medicinal use in India. "It is successfully used in jaundice, ascites, anasarca, scanty urine and internal inflammations," (Dutt). It is also used as a diuretic in gonorrhoea. It has always been used by the French and the Portuguese in all the above mentioned troubles with a marked success. If it is taken in sufficient doses it acts as a powerful emetic.

The chemical analysis of the plant was undertaken in 1910, and the following results were obtained[1] :—

It contains (1) a sulphate of a body alkaloidal in nature, (2) an oily amorphous mass of the nature of fat (probably) and (3) sulphates, chlorides and traces of nitrates and chlorates from the ash. The amount of alkaloidal body is very small.

Dymock[2] mentions all the above properties of the drug and says that the whole plant was used for a chemical analysis and with the exception of minute traces of a principle soluble in ether and affording reactions with alkaloidal reagents nothing of interest was detected. No principle reacting with ferric salts was present. The above represents the work that has been done on this plant. On account of its high medicinal value the present authors were tempted to put it to a more systematic chemical examination. Unfortunately the so called 'alkaloidal' body supposed to be present in the plant could not be detected and hence the claims of the previous workers were not substantiated. However, as a result of investigation, it was found that the plant contains about ·05% of a crystalline acid, 1% of potassium nitrate, 1·2% of a brown amorphous mass consisting mostly of tannins and phlobaphenes and reducing sugars (mainly glucose).

EXPERIMENTAL

The plant was collected from the neighbourhood in the month of October and dried for a month in the shade. The dried plant was then finely crushed in an iron mortar, and when burnt completely in a porcelain dish left 10·2% of a dirty white ash. The ash contained 74·8% of water insoluble and 25·2% of water soluble inorganic matter. The following elements and radicals were detected in the ash:—

Potassium, Sodium (traces), Magnesium, Calcium, Nitrates, Phosphates, Carbonates. Silica, and Sulphates (traces).

In order to ascertain the general characters of the soluble portion of the plant samples of finely powdered material were exhaustively extracted in a Soxhlet's apparatus using various solvents, whereby the following amounts of extracts dried at 100° were obtained.

1. *Alcoholic Extract:* 13·5%. Dark yellowish green sticky mass smelling strongly of chlorophyll, also containing some crystalline mass. It gave a precipitate with lead acetate and silver nitrate, reduced Fehlings' solution. Gave a green colouration with ferric chloride, no reaction with Meyer's reagent (for alkaloids).

2. *Petroleum Ether Extract:* 10·3%. A syrupy extract was obtained which contained nothing but chlorophyll.

3. *Benzene Extract:* 5·0%. Properties similar to the above.

4. *Acetone Extract:* 9·4%. A sticky mass consisting of chlorophyll and some crystalline matter was obtained. Properties similar to the alcoholic extract.

5. *Aqueous Extract:* 2·9%. The extract was of a syrupy nature. It reduced Fehling's solution easily. Gave a bottle green colouration with neutral ferric chloride, a brown precipitate with Meyer's Reagent. Gave a negative test for glucosides.

A preliminary examination was made with 200 gms. of the powdered drug The alcoholic extract was treated with a little hydrochloric acid and tested for alkaloids with various reagents. The following precipitates or colourations were obtained:—

Phospho-molybdic acid	...	A dark blue precipitate.	
Phospho-tungstic acid	...	A brown precipitate.	
Fröhdes' reagent	...	Do	Do
Mandellin's reagent	...	Brown colouration.	
Dragendrof's reagent	...	Do	Do
Meyer's reagent	...	Do	Do

Picric acid	... No change.
Erdmann's reagent	... A brown colouration.
Sodium-bi-carbonate	... No change.

In order to confirm the above result, another 200 gms. of the finely powdered material were boiled with Prollius' solvent[3], *i.e.*, a mixture containing 88 parts ether, 4 parts ammonia (d. 1:2) and 8 parts alcohol for two hours. The clear supernatant liquid was filtered off and sulphuric acid was added, the thick fluid was separated and treated with alkaloidal reagents with negative results.

In order to get a complete analysis 8 Kg. of the powdered drug was exhaustively extracted with boiling alcohol in a big extraction flask. The extract was filtered hot and concentrated to a small volume. A dirty brown crystalline mass separated on cooling which was greatly contaminated with chlorophyll. The mass then became dirty brown in appearance. This was crystallised from alcohol after refluxing with animal charcoal. Finally it was recrystallised from a mixture of acetone and alcohol, when brown microcrystalline flakes were obtained melting at 108-109°C (decomp.). It was soluble in hot alcohol, hot acetone, chloroform, ether, and insoluble in water, petroleum ether, benzene, acetic acid, phenol and dilute mineral acids. It gave a darkish green precipitate with neutral ferric chloride, and a white precipitate with alcoholic lead acetate. It dissolved in alkalies and burned with a non-smoky flame. (Found: C, 64·61, 64·40; H; 9·49, 9·54; M. W. (decomposition of lead salt) 180, 184. $C_{10}H_{18}O_3$ requires C, 64·5; H, 9·7% and M. W. 186.)

It is proposed to call this acid *Bœrhaavic acid* in order to represent the generic name of the plant from which it is derived.

This acid was very inert since an attempt to prepare an acetyl derivative and an oxime proved unsuccessful. The only derivatives that could be prepared from it were the lead salt and a di-bromo-bromide.

Lead Salt.—1 gm. of the acid was dissolved in hot alcohol and an alcoholic solution of lead acetate was added drop by drop when a flocculant white precipitate of the lead salt was obtained. This was filtered off, washed with alcohol and dried. It formed brownish white plates. (Found: Pb, 53·19, 53·4; $C_{10}H_{17}O_2Pb'$ requires, Pb, 53·0%)

Di-bromo-bœrhaavic bromide.—0·5 g. of the acid was dissolved in chloroform and a solution of bromine in chloroform was added drop by drop till bromine was in slight excess. It was then warmed on a water-bath, and the excess of bromine removed by means of an aqueous solution of thiosulphate. On distilling back the chloroform, the bromo-derivative was obtained as an yellowish brown amorphous mass. This

was crystallised from alcohol, when it melted sharp at 73°C. In this reaction some elimination of hydro-bromic acid was also detected. (Found: Bromine, 56·2 %. $C_{10}H_{17}O_3Br_3$ requires Br, 56·4%).

The mother liquor after the separation of the above acid on standing deposited, white crystalline needles which were filtered and identified to be inorganic in nature. These on examination proved to be pure potassium nitrate, the presence of which in the plant greatly accounts for its medicinal value as a diuretic. *Cf.* Pendse and Dutt[4], examination of *Solanum Xanthocarpum.*

The alcoholic mother liquor was then concentrated to a small bulk and dried in open for about a month. It smelt largely of chlorophyll and sugars. It was refluxed with benzene to remove chlorophyll till benzene became colourless. The stuff was then dissolved in hot alcohol and an alcoholic solution of lead acetate was added when a heavy yellow precipitate was obtained. This was filtered and washed. The filtrate on treatment with basic lead acetate gave another precipitate which was also filtered and washed. These precipitates were suspended in alcohol and water respectively and decomposed by hydrogen sulphide. The filtrates on the removal of the lead sulphide were evaporated, whereby amorphous dark brown hygroscopic products were obtained consisting mostly of tannins and phlobaphenes, since they gave a bottle green colouration with ferric chloride and dissolved in caustic soda being re-precipitated by acids. It also formed a red colouration with potassium ferricyanide and ammonia.

A portion of the filtrate after the removal of the lead salt, was freed from lead. It reduced Fehlings solution easily. In order to identify the sugar a portion was treated with acetic acid and phenylhydrazine, when on warming an osazone was prepared (M.P. 203°C) showing the presence of glucose in the plant.

One of the authors (R. R. A.) wishes to express his indebtedness to the "Kanta Prasad Research Trust" of the Allahabad University for a scholarship, which enabled him to take part in the investigation.

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SYNTHESIS OF SUBSTITUTED CINCHONINIC ACIDS THROUGH THE KNŒVENAGEL CATALYSTS

By K. Madhusudanan Pandalai

Chemistry Department, Andhra University

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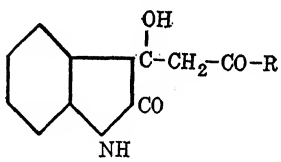
The Pfitzinger method[1] of synthesis of substituted cinchoninic acids like cinchophen from isatin and methyl ketones with strong potash as condensing agent affords a convenient method when no alkali sensitive groups are present. Recently Lindwall and Maclennan[2], Braude and Lindwall[3] attempted to use the more gentle catalysts namely diethylamine, piperidine and ammonia suggested by Knœvenagel[4] in place of the concentrated alkali. With these Knœvenagel catalysts no substituted cinchoninic acids were obtained from an alcoholic solution of isatin and methyl ketone but compounds of the type (I) were formed.

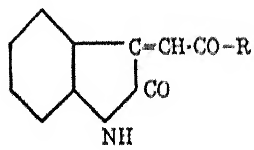


(I)

Thus with isatin and acetophenone no cinchophen was formed, but only 3-hydroxy-3-phenacyl oxindole, and isatin and acetone giving no 2-methyl-quinoline-4 acid but only 3-hydroxy-3-acetonyl oxindole. Compounds of the type (I) decompose on heating near their melting points back into isatin and the methyl ketone. The same decompositions take place when products of the type (I) are warmed with potash solution for a short time. The product (I), therefore, is evidently only an addition compound and the decomposition an example of a reversed aldol-like condensation. When these compounds are, however, subjected to the drastic Pfitzinger conditions (heating it with strong alkali for a very long time) they yield the 2-substituted-quinoline-4-carboxylic acids. Lindwall and collaborators hold the view that this formation of the quinoline derivative from (I) did not occur through any molecular rearrangement

but rather through its decomposition back into isatin and the methyl ketone followed by condensation of the two according to the usual Pfitzinger mechanism.

Treatment of compounds of the type (I) with dilute mineral acid resulted in its dehydration to yield the 3-substituted unsaturated oxindoles (II).



(II)

It might be expected that compounds of type (II) upon ring opening through hydrolysis would form the quinoline derivative as a result of subsequent reaction of the ketone carbonyl with the nuclear amino group. But this expectation could not be realised experimentally by those authors (*loc cit.*) who suggest the possibility of compounds of the type (II) being *trans* isomers and their consequent failure to rearrange to give the quinoline derivatives. These results have lead Lindwall and coworkers to suggest also that compounds of the type (I) and the 3-substituted unsaturated oxindoles (II) cannot possibly be intermediates in the Pfitzinger synthesis of substituted cinchoninic acids from isatin and methyl ketones. From this it is clear that Lindwall and collaborators could not obtain the substituted cinchoninic acids by the use of the Knœvenagel catalysts. These points have not yet been investigated completely. The present work was, therefore, undertaken to examine the conclusions of Lindwall and coworkers and see if it is possible to get substituted cinchoninic acids using the mild Knœvenagel catalysts instead of the drastic Pfitzinger one. The substituted cinchoninic acids are therapeutically important for their antineuralgic effects and as remedy for gout, and as an analgesic in the treatment of sciatica.

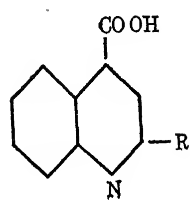
In cyclic compounds which are capable of giving rise to *cis-trans* isomerism it is well known that intramolecular reactions will take place more readily when the reacting groups are in the *cis* position to one another than when they are in the *trans* position and in certain cases it may be easy to produce in the *cis* compound a reaction which is difficult or impossible in the case of a *trans* isomer. This is exactly the position of the supposed intermediates (compounds of type (II)) in the Pfitzinger reaction. Lindwall and coworkers found that compounds of type (II) cannot rearrange to give substituted cinchoninic acids possibly because

they are *trans* isomers. It is very likely that the corresponding *cis* isomers would readily yield the desired products. This has actually been found to be the case. For effecting the geometrical inversion the method of Stoermer[5] was conveniently adopted. This consists in exposing the *trans* derivative in alcoholic solution to light rich in ultraviolet rays such as that furnished by the mercury vapour lamp. The *cis* modification thus obtained was found to give substituted cinchoninic acids on gentle warming with dilute potash. Thus it appears that geometrical inversion is necessary before (II) can rearrange to the corresponding cinchoninic acids. Since rearrangement is possible only in the *cis* form and since compound (II) after geometrical inversion yielded readily the desired product, *viz.*, the 2-substituted-quinoline-4 carboxylic acid it was reasonably presumed that compounds of type (II) are *trans* modifications and attempts at determining the actual configuration were dispensed with. Thus it is possible to get from isatin and methyl ketone the substituted cinchoninic acids, even by using the Knœvenagel catalysts. The synthesis may be expressed in the following stages:

The beta carbonyl of the isatin condenses with the methyl group of the methyl ketone giving the addition compound. (I)

Beta-substituted isatins like isatin-beta-anil, isatin-beta-bis-piperidyl, etc., give no products with these catalysts indicating that alpha carbonyl of the isatin does not take part in the condensation. N-methyl-isatin condenses with methyl ketones to give products similar in chemical properties showing that linkage of the methylketone to nitrogen of the isatin is an impossibility. Knœvenagel catalysts cannot compare themselves with the Pfitzinger condensing agent in these condensations, because isatin does not hydrolyse under these conditions but gives only the addition product (I) shown above. This product (I) is highly unstable and gives rise to (II) on treatment with dilute mineral acid by dehydration

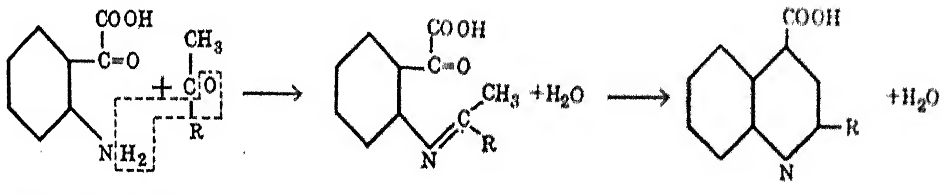
These unsaturated compounds are *trans* isomers and require geometrical inversion to give the *cis* compounds. These *cis* varieties on treatment with dilute-alkali and gentle warming gave (III) the 2-substituted cinchoninic acids.



(III)

It is therefore clear that the synthesis using Knoevenagel catalysts involves the use of several reagents in the different stages unlike in the Pfitzinger reaction where the formation of the cinchoninic acids is direct. This is usually conceived to take place as follows.

The isatin hydrolysis into orthoamino-benzoyl-formic acid must be the first stage. Then the ketone carbonyl of the methyl ketone reacts with the amino group of the orthoamino benzoyl formic acid which must be the second stage. The alpha carbonyl of the isatin has been destroyed, of course, during the hydrolysis. In the third stage a molecule of water is eliminated from the carbonyl of the benzoyl-formic acid and of the methyl in the methyl ketone effecting thus a ring closure when the quinoline derivative is formed. This is usually accomplished at a single stretch, *viz.*, heating the reaction mixture from eight to ten hours.



orthoamino-benzoyl-formic acid.

It was noticed that sun-light cannot bring about the geometrical inversion of the 3-substituted unsaturated oxindoles.

Experimental

A. (*i*) Condensation of isatin and acetone. 3-acetonyl-3-hydroxy oxindole (I). A mixture of 25 grs. of isatin, 139 grs. of acetone and 13·9 grs. of diethyl amine were allowed to stand overnight with a trace of ammonia (crystals were not obtained as reported by Lindwall and others). This was refluxed on a water bath for two hours. The hot liquid poured into ice-cold water when a solid mass was precipitated. Acetone was the best solvent for crystallisation; agreed in melting point and other properties with that obtained by Lindwall. (M. P. 166-167°C) yield 75%. Piperidine gave 60% and ammonia gave 73% yields.

(*ii*) 3-acetonylidene-oxindole (II), compare Lindwall, (*loc. cit.*). Red needles from ethyl alcohol (M. P. 168-171°C). Lindwall, M. P. 169°C. 40% yield.

(*iii*) 2-methyl-quinoline-4-carboxylic acid (aniluvitonic acid) from isatin and acetone.

(*a*) Using Pfitzinger method.
25 grs. of isatin, 139 grs. of acetone, and 20 cc. of 33% potash heated together for $9\frac{1}{2}$ hours and acidified with dilute hydrochloric acid to precipitate a light brown substance. Boiling water was used to crystallise this which was later on identified to be 2-methyl-quinoline-4-acid.

(*b*) Using Knœvenagel catalysts and subsequent geometrical inversion. 5 grs. of 3-acetonylidene oxindole was dissolved in 45 cc. of absolute alcohol and this solution placed in a cubical quartz cell. This was exposed to ultraviolet light from a mercury vapour lamp (with the proper condensing arrangements, etc.) for about 5 to 6 hours.

A crystalline solid gradually settled. This was separated and recrystallised from alcohol. It melted at 162°C. This was colourless and microcrystalline, on keeping it became slowly red (probably changing back into the more stable *trans* isomer). This powder was dissolved in 33% potash made just alcoholic and gently warmed in a water bath. On acidification with dilute hydrochloric acid a solid precipitated which was crystallised and identified to be 2-methyl-cinchoninic acid (yield 1·9 g.) M. P. 242°C (compare Baeyer[6]). (Note.—Mulliken gives 240-241°C as one melting point and 246°C as another. Lindwall and coworkers do not mention the melting point of the product.)

(*iv*) Action of alkali on (I). A specimen of (I) on heating for 8 hours with sufficient excess of potassium hydroxide gave on acidification a precipitate which was identified to be 2-methyl-quinoline-4-acid.

(*v*) Action of alkali on (II). A specimen of (II) on heating for 10 hrs. with potassium hydroxide in excess and acidification gave a dark tarry oil from which no substance could be purified.

(*vi*) Action of sunlight on (II). A specimen of (II) on exposure to sunlight in (*a*) alkali solution (*b*) in alcoholic solution gave dark masses from which no substance could be purified.

B. (*i*) Condensation of isatin and acetophenone.
3-phenacyl-3-hydroxy-oxindole.

Isatin 5 parts, acetophenone 4 parts, caustic potash (33%) 20 parts. (Pfitzinger) or 10 parts of a basic catalyst (Knœvenagel) as diethyl amine, piperidine, ammonia, etc., were treated according to the procedure described by Lindwall (*loc. cit.*). Caustic potash gave the optimum yield of 80% (M. P. 169 – 172°C) with decomposition.

(*ii*) 3-phenacylidene oxindole (compare Lindwall and others) M. P. 193-194°C.

(*iii*) cinchophen (2-phenyl-quinoline-4-carboxylic acid) from isatin and acetophenone.

(*a*) Using Pfitzinger method. Compare method A (*iii*) (*a*) Needles melting at 210°C.

NOTE.—(Pfitzinger gives its melting point as 208-209°C (*loc. cit.*). Boehm and Bournot[7] report it to be 212-213°C. Lindwall and others do not record their melting point in their paper but only say that they identified it by mixed melting point.)

(*b*) Using Knœvenagel catalysts and subsequent geometrical inversion (of 2-phenacylidene oxindole).
Compare A (*iii*) (*b*)
(The *cis* modification melted at 193°C yield 3 g. from 5 g. of (II). It was microcrystalline.)

(*iv*) Compare A (*iv*) melting point of product 210°C.

(*v*) Compare A (*v*) above.

(*vi*) Compare A (*vi*) above.

A few more substituted cinchoninic acids are being studied from this point of view. My sincere thanks are due to Mr. G. Gopala Rao, M.Sc., A.I.C., for the kind interest he has taken in my work.

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PHOTOSYNTHESIS OF FORMALDEHYDE FROM 'NASCENT CARBON DIOXIDE' *IN VITRO* AND THE IMPORTANCE OF RESPIRATION IN PHOTOSYNTHESIS

By Atma Ram

Chemistry Department, University of Allahabad

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In several publications[1], from these laboratories we have shown that formaldehyde can be actually obtained from the photochemical reduction of carbon dioxide in aqueous solutions in the presence of various inorganic and organic photocatalysts. In the absence of a photocatalyst, the yield of formaldehyde obtained in these experiments, is very small. Further, it has been shown that dilute solutions of alkali bicarbonates when exposed to sunlight, yield small amounts of formaldehyde even in the absence of a photocatalyst. Recently[2], we have been able to obtain greater yields of formaldehyde by exposing to sunlight bicarbonate solutions in the presence of metals like magnesium, iron, cerium, etc., and that the reaction is accelerated by light.

In all these experiments carbon dioxide which has been used is in the inactive state. The reaction

$$CO_2 + H_2O = HCHO + O_2 - 112{,}000 \text{ cals.}$$

is a highly endothermal one and requires a wavelength of about 2550 Å. The fact that bicarbonate solutions yield formaldehyde on exposure to sunlight even in the absence of photocatalysts, clearly points out that there is some difference in ordinary carbon dioxide and that produced in the photodecomposition of bicarbonates. The only difference that seems to exist between these two varieties of carbon dioxide is that carbon dioxide obtained from the decomposition of bicarbonates is in the nascent state. If that is so, then nascent carbon dioxide should yield formaldehyde more readily than the ordinary variety without using any photocatalyst, whenever it can be produced in

the presence of sunlight. This has already been confirmed by the author[3]. It has been shown that formaldehyde can be easily obtained from nascent carbon dioxide prepared by the interaction of carbonates of barium, calcium, strontium and sodium with hydrochloric acid, when the whole system is exposed to light in the absence of any coloured substance.

Thus it seems pretty certain that nascent carbon dioxide can be photochemically reduced to formaldehyde more readily than ordinary carbon dioxide. In all these experiments nascent carbon dioxide has been generated from inorganic materials. Is there a possibility of generating carbon dioxide in a nascent state from organic substances and utilising it for the photosynthesis of formaldehyde? A serious attempt has been made along this line and in the following pages I shall try to give an account of the experiments that have been carried on to test this point.

It is well known that many organic compounds yield formaldehyde on exposure to sunlight and air. The action of light on organic substances has been studied by many observers. The important work on this subject is that of Usher and Priestley[4], Jörgensen and Kidd[5], Warner[6]. These authors have made an extensive investigation on the generation of formaldehyde when films of chlorophyll are exposed to sunlight in the presence and absence of air. The origin of formaldehyde detected when aqueous solutions or suspensions of organic substances are exposed to sunlight and air, has not yet been satisfactorily explained.

With a view to the elucidation of the mechanism of generation of formaldehyde from the photodecomposition of organic substances and also to see whether there is any relation between the production of formaldehyde from organic substances and photosynthesis by the plants a systematic investigation on the formation of formaldehyde by exposing solutions or suspensions of various organic substances to sunlight and air has been undertaken.

Experimental

Dilute solutions of the organic substances were exposed in open beakers to sunlight for about six hours, and after the desired exposure the solutions were distilled and the amount of formaldehyde estimated by the iodine method or colorimetrically by the Schryver's reagent using the Dubosoq type of colorimeter. As a result of the estimation of the amount of formaldehyde obtained from the photodecomposition of organic substances they have been divided into the following three groups.

Table I

GROUP A

The amount of formaldehyde obtained from 100 cc. of the solution exposed, varying from 0·0007 to 0·0015 g.

Experiment.	Time of exposure in hours.	Amount of formaldehyde in grms. per 100 cc. of the solution exposed.
1. 50 cc. dilute acetic acid containing 5 cc. glacial acetic acid.	6 hours.	0·0015
2. 50 cc. of 1% solution of glycine	do.	0·0016
3. 50 cc. of 0 02% solution of methyl violet ...	do.	0·0012
4. 50 cc. of 0·02% solution of malachite green ...	do.	0·001
5. 50 cc. of 0·02% solution of victoria blue	do.	0·001
6. 50 cc. of 0·02 solution of methylene blue ...	do.	0·0008
7. 50 cc. of 3% solution of citric acid	do.	0·001
8. 50 cc. of 5% solution of lactic acid	do.	0·001
9. 50 cc. of a solution of guaiacol (absolute) containing 1 cc.	do.	0·0009
10. 50 cc. of water containing 0·05 gr. of pure chlorophyll	do.	0·0007
11. 50 cc. of 5% solution of monochloracetic acid ...	do.	0·0015

Table II

GROUP B

The amount of formaldehyde obtained from 100 cc. of the solution exposed, varying from 0·0001 to 0·0006 g.

Experiment.	Time of exposure.	Amount of formaldehyde in grms. obtained from 100 cc. of the solution exposed.
1. 50 cc. of a dilute solution of pyruvic acid	6 hours	0·0004 g.
2. 50 cc. of 2% tartaric acid ...	do	0·0002 g.
3. 50 cc. of 0·02% phloxine ...	do.	0·0001 g.
4. 50 cc. of 0·02% aurine ...	do.	0·00016 g.
5. 50 cc. 0·02% solution of gallomine blue	do.	0·0003 g.
6. 50 cc. of 0·02% solution of 4:5 dihydroxy fluran ...	do.	0·0003 g.
7. 50 cc. of 1% solution of levulose	do.	0·0006 g.
8. 50 cc. of 5% solution of galactose	do.	0·0003 g.
9. 50 cc. of 5% solution of butyric acid	do.	0·00012 g.
10. 50 cc. of 5% solution of propionic acid ...	do.	0·00012 g.

Table III

GROUP C

The amount of formaldehyde obtained from 100 cc. of the solution varying from 0·000015 to 0·0001 g.

Experiment.	Time of exposure in hours.	Amount of formaldehyde in grams. obtained from 100 cc. of the solution exposed.
1. 50 cc. of a solution of sodium oleate, M/100 ...	6 hours	0·0001
2. 50 cc. sodium butyrate solution M/100	do.	0·00015
3. 50 cc. of a 0 02% solution of congo red	do.	0·00003
4. 50 cc. of a 0·02% solution of safranine	do.	0·00003
5. 50 cc. of 0·02% solution of oxalic acid	do.	0·00004
6. 50 cc. of a solution of starch 1%	do.	0·00003
7. 50 cc of a solution of inulin, 1%	do.	0·00003
8. 50 cc. of a solution of glucose, 1%	do.	0·00005
9. 50 cc of a solution of cane sugar, 1%	do.	0·00004
10. 50 cc. of a 0·02% solution of acridine yellow ...	do.	0·00006
11. 50 cc. of a 0·02% solution of orthochrome	do.	0·00007
12. 50 cc. of 3% solution of gelatine	do.	0·00006

Experiment	Time of exposure in hours	Amount of formaldehyde in grams obtained from 100 cc. of the solution exposed.
13. 50 cc. of 1% solution of glutamic acid ...	6 hours	0·00004
14. 50 cc. of a 1% solution of arginine	do.	0·00008
15. 50 cc. of a 0·02% solution of pinacyanole	do.	0·00005
16. 50 cc. of a 0·02% solution of benzopurpurin ...	do.	0·00004
17. 50 cc. of 0·02% solution of pinachrome	do.	0·00005
18. 50 cc. of a solution of alanine, 1%	do.	0·00004
19. 50 cc. of a 1% solution of histidine	do.	0·00005
20. 50 cc. of 1% solution of asparatic acid ...	do.	0·000055

Discussion

It has already been pointed out that aqueous solutions of bicarbonates of alkali metals give formaldehyde on exposure to sunlight. It is well known that bicarbonate solutions decompose into the corresponding carbonate and carbon dioxide. Thus carbon dioxide which is generated in the decomposition of bicarbonates is in the nascent state and hence is energy rich. The extra amount of energy which it possesses by virtue of being a nascent molecule may be partly utilised for its activation in order to be reduced to formaldehyde. To obtain formaldehyde from bicarbonate solutions it is not necessary to add any photosensitiser as with ordinary carbon dioxide. Thus it is clear that nascent carbon dioxide is more readily reduced to formaldehyde than ordinary carbon dioxide and the reason of this peculiar behaviour

is to be ascribed to the extra amount of energy possessed by the former. That formaldehyde is also obtained from carbon dioxide generated by the interaction of colourless carbonates like those of barium, calcium, etc., and mineral acids without the addition of any photosensitiser also supports the view that nascent carbon dioxide is more capable of yielding formaldehyde on its photochemical reduction than the ordinary one, simply because in these experiments too, nascent carbon dioxide is responsible for the production of formaldehyde.

What is the origin of formaldehyde produced from the photodecomposition of photo-oxidation of organic compounds? From the list of substances given in tables I, II & III it will be seen that they belong to different classes of organic compounds and hence no general rule as to the production of formaldehyde from these substances can be formulated. The organic compounds mentioned above have been classified into three groups according to the amount of formaldehyde obtained in their photodecomposition or photo-oxidation and not with respect to their chemical nature.

In the first group, the substances on exposure to light and air yield formaldehyde very readily and in sufficient amounts. By carrying on experiments in the dark and in the absence of air we have observed that no formaldehyde is formed in the dark or in the absence of air. When solutions of these substances are exposed to light and air, formaldehyde is formed mainly as a direct product of photo-oxidation. In a recent communication[7], it has been advanced by the author that the first stage in photosynthesis by the plants and *in vitro* is the photolysis of water molecules into H and OH. Thus in the case of glycine the following change may take place in the presence of air and light:

(1) $H_2O \rightarrow H + OH$

(2) $NH_2.CH_2COOH + H.OH = NH_3 + CH_2(OH)COOH$

(3) $CH_2(OH)COOH + O = CO_2 + H_2O + HCHO$

and in the case of acetic acid;

(1) $CH_3COOH + H.OH = CH_3OH + HCOOH$

(2) $CH_3OH + O = H_2O + HCHO$

This mechanism can very easily explain the production of formaldehyde from acetic acid, by the oxidation of methyl alcohol. The production of formaldehyde from the second class of substances may not be entirely due to the direct photo-oxidation of the substance, but due to the joint action of oxidation and photosynthesis (explained later).

Mechanism of the production of formaldehyde from substances of the third group and the Formaldehyde Theory of Carbon Assimilation.—It is well known that carbon dioxide and water are always generated whenever an organic compound is slowly oxidised, and an appreciable amount of energy is simultaneously set free. For instance, in the case of glucose,

$$C_6H_{12}O_6+3O_2=6H_2O+6CO_2+E.$$

The carbon dioxide so produced, at the moment of its generation will be in the nascent state or energy rich, since it is being produced in the presence of a large amount of energy. The reaction,

$$CO_2+H_2O=HCHO+O_2-110{,}000 \text{ cals.}$$

is a highly endothermal one, and very difficult to be brought about in the usual way. From the thermodynamic considerations the above reaction demands a minimum frequency of 1.175×10^5 *i.e.*, a wavelength of 2552 Å. Thus it is clear that without the help of an efficient photo-catalyst, it is difficult to obtain formaldehyde from the photochemical reduction of carbonic acid. But all these experiments have been carried on in the absence of any photosensitiser, hence a mode of activation of ordinary carbon dioxide should be sought for, in order to explain photosynthesis of formaldehyde from carbonic acid.

The greater efficiency of nascent carbon dioxide which is energy rich, must be attributed to the higher energy content (which it possesses as a freshly generated molecule), than one of ordinary carbon dioxide. Thus a molecule which is already endowed with energy, should require less energy for its activation than one which is deficient in it and hence nascent carbon dioxide can be photochemically reduced to formaldehyde more readily than ordinary carbon dioxide. Since nascent carbon dioxide and water are being produced from organic substance in the presence of sunlight and it is very likely that under the action of sun's rays they should combine even in the absence of any photosensitiser. It is this formaldehyde which is being detected in the photo-oxidation of organic substance.

From these considerations it appears that the source of formaldehyde generated in the photo-oxidation of substances like oxalic acid, acridine yellow, alanine, cane sugar, etc., is not the direct photo-oxidation of the latter, but its photosynthesis from the nascent carbon dioxide and water generated during photo-oxidation of the organic substances. It is due to this reason that formaldehyde is detected in the photo-oxidation of

organic substances because the two ingredients are almost universally generated during the oxidation of the former. From comparative experiments it has been found that a solution of alanine is decomposed to a greater extent than a solution of glycine of the same concentration, though the amount of formaldehyde formed from the latter is at least twenty times as much as from alanine. This is simply due to the reason that in the case of glycine, formaldehyde is obtained as a direct product of photo-oxidation of the former whilst with alanine it is obtained photosynthetically from energy rich carbon dioxide generated in the oxidation.

In this connection it will be worthwhile noting that formaldehyde is obtained even in the dark by treating carbonic acid or bicarbonate solutions with metals like magnesium, cerium, iron, etc. Recently we have been able to show that small amounts of formaldehyde are obtained by treating bicarbonate solutions with yellow phosphorus. In these cases the amount of formaldehyde formed is greater in light than in the dark. It appears that the energy rich hydrogen produced by the action of metals on water is capable of reducing the bicarbonate ion or carbonic acid to formaldehyde even in the dark, aided by the energy produced by the reaction of the metal on carbonic acid and bicarbonate solutions.

From the results of quantitative experiments carried on in this connection it has been found that the ratio of the mol. of the substance decomposed to the mols. of HCHO formed is approximately unity in the case of substances of the first group (acetic acid and glycine), whilst with the substances of the third group this value has been found to vary from 35 to 300. These results afford another argument in favour of the view advanced here, because the efficiency of the photosynthetic process is very low and hence all the molecules of carbon dioxide in the photo-oxidation cannot take part in the reaction; that is why in this case the ratio comes out to be very high.

From this theory it is expected that the yield of formaldehyde should be greater, the greater the energy generated in the oxidation of the organic substance. It has been experimentally found that the fats give more formaldehyde on their photo-oxidation than carbohydrates or the proteins, though the fats are oxidised less than the other two classes of substances. This is due to the fact that fats give out more energy on their oxidation than the carbohydrates or the proteins, and thus causing the reduction of carbonic acid to a greater extent.

The production of formaldehyde in the oxidation of organic sub-stances has been regarded by several authors as an argument against the formaldehyde theory of photosynthesis, but from the views advanced here it will be seen that this fact instead of being against the theory lends appreciable support to this view.

Importance of Respiration in Photosynthesis.—It seems very likely that the energy generated in the photo-oxidation of organic compounds supplies a part of the energy required for the photosynthesis of formaldehyde from carbon-dioxide obtained from the oxidation of organic substances and that is why formaldehyde can be more readily detected in the photo-oxidation of organic compounds which liberate energy in their oxidation, than in the case where ordinary carbon dioxide or bicarbonate solution are exposed to sunlight. Many plant physiologists have tried to connect respiration with photosynthesis, but the mechanism in which these two fundamental processes going on in the plants may be related is not yet clearly understood. Thus, Spoehr (*Photosynthesis*, 1926, p. 143) states, "Also, the light green varieties have lower rate of respiration than the normal plants, though there is no direct parallelism between respiration and chlorophyll content. A closer relationship seems to exist between the rate of photosynthesis and that of respiration." A relation between respiration and photosynthesis has been observed by Miss Henrici also (Inaug. Diss, Basel 1918). She has observed in the study of alpine and lowland plants that those plants which had a high photosynthetic rate also have a high rate of respiration and *vice versa*.

In previous pages it has been shown that the greater the energy evolved in the oxidation of the substance the greater is the amount of formaldehyde formed. Instead of the oxidation of an organic substance which is the source of energy here, any other suitable exothermal reaction will suffice to increase the yield of formaldehyde from the photo-oxidation of organic substance. Since respiration is the oxidation of carbohydrates and other food materials of the plant, it follows from the viewpoints advanced here, that the greater the rate of respiration, the greater is the possibility of a supply of energy for photosynthetic purposes. Thus respiration is directly related to photosynthesis and the way in which it is related to the latter, is the supply of a part of energy (evolved during respiration) which is required for photosynthesis.

Just as photosynthesis *in vitro*, *i.e.*, the formation of formaldehyde by exposing solutions of carbonic acid to sunlight, is exceedingly difficult in the absence of an exothermal reaction taking place along with

it, carbon assimilation in plants is also practically impossible in the absence of a supply of energy available from respiration. It will be interesting to note here that the formation of urea in the body is greatly enhanced by respiration. It appears, therefore, that besides light, carbon dioxide, moisture and chlorophyll, energy from respiration is also necessary for photosynthesis, and that is why those plants which respire well also photosynthesise well.

The author is carrying on these experiments further and a comprehensive theory of photosynthesis based on these observations will be given in a subsequent paper. My best thanks are due to Professor N. R. Dhar, for the keen interest that he has taken during the progress of this work.

Summary

1. When solutions of organic substances like acetic acid, citric acid glycine, malic acid, lactic acid, glycogen, acetone, etc., are exposed to sunlight and air, formaldehyde is readily detected.

2. It has been shown that the aqueous solutions of the dyes like malachite green, methyl violet, methylene blue, etc., form formaldehyde readily on photo-oxidation. These dyes also behave as antiseptics, and it is very likely that the production of formaldehyde from these substances when exposed to light, may be the real cause of the antiseptic action.

3. Tartaric acid, butyric acid, propionic acid, and some other dyes form smaller quantities of formaldehyde while oxalic acid, formic acid glucose, cane sugar, starch, histidine, etc., produce very small amounts of formaldehyde from photo-oxidation.

4 It is believed that formaldehyde formed in the first group of substances is a direct product of the photo-oxidation, whilst with oxalic acid, starch, glucose, etc., formaldehyde is obtained from the photosynthesis of energy rich carbon dioxide and water produced during photo-oxidation.

5. It seems probable that the energy generated in the photooxidation of organic compounds supplies a part of the energy required for the photo-formation of formaldehyde. In nature, photosynthesis that takes place in plants is aided by the energy obtained in plant respiration.

6 In the plant kingdom, the process of respiration and photosynthesis seem to be very intimately connected, because photosynthesis cannot proceed without the energy available from respiration for the partial activation of carbon dioxide and water vapour.

7. Formaldehyde can be obtained in a much easier way from carbonic acid or bicarbonate solutions on exposure to sunlight when a suitable exothermal reaction is taking place in the system along with the photosynthetic reaction.

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ON AMPHISTOME PARASITES OF SHEEP AND GOAT FROM ALLAHABAD

By K. R. Harshey

Zoology Department, University of Allahabad

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Introduction

The table given below gives in brief the analysis of most of the contributors of the Amphistomata Group in the last thirty years:

Table 1

No.	Host.	Subfamilies.	Contributors.
1.	Fishes.	*Diplodiscinae.*	Fishes. Daday, 1901; Nicoll, 1915; Goto and Mutsudaira, 1918. Travassos. Artigas, and Pereira, 1928; Vaz, 1932.
2.	Amphibia.	*Schizamphistominae.*	Amphibia. Cohn 1904; Johnston, 1912; Stunkard, 1917; Mac Callum, 1917.
3.	Reptiles.	*Cladorchinae*	Reptiles. Cohn 1904; Looss, 1912; Stunkard, 1917.
4.	Birds.	*Zygocotylinae*	Fischoeder, Stunkard, 1917.
5.	Mammals.	*Paramphistominae* *Gastrodiscinae* *Balanorchinae* *Pfenderinae*	Fischoeder, 1901-03; Leiper, 1908-10; Stiles and Goldberger, 1910; Mac Callum, 1917; Maplestone, 1923; Fukui, 1926-1929.

The contributions to this group in the 19th century have been those of Rudolphi, 1809; Diesing, 1835; Creplin, 1839-47; Sonsino, 1876-95; Poirier, 1882-83; Monticelli 1888. In the last thirty years the most important contributions to this group have been those of Fischoeder, 1901-03; Cohn, 1904; Leiper, 1908-10; Stiles and Goldberger, 1910; Odhner, 1911; Looss, 1912; Johnston. 1912; Stunkard, 1917; Maplestone, 1923; Fukui, 1926-29, Travassos, Artigas and Pereira, 1928 and Vaz, 1932. But we owe

our present knowledge of the classification of the group to the efforts of Monticelli, 1888; Braun, 1889-93; Fischoeder, 1901-03; Stiles and Goldberger, 1910; Stunkard, 1917 and 1925; Maplestone, 1923 and Fukui, 1929.

In 1892 Monticelli for the first time divided the family *Paramphistomidae* into two subfamilies *Gastrodiscinae* for *Gastrodiscus* and *Cladorchinae* for the remaining amphistomes.

Fischoeder (1901) further divided the subfamily *Cladorchinae* and created one more subfamily *Paramphistominae* to include the amphistomes of mammals, characterised by the presence of lobed testes, absence of paired oral evagination and cirrus sac; whereas the *Cladorchinae* distinguished from it by the presence of branched testes, paired oral evaginations and cirrus sac. He dropped the subfamily *Gastrodiscinae* and included the genus in the *Cladorchinae.* Cohn (1904) created the subfamily *Diplodiscinae* for *Diplodiscus*, *Opisthodiscus*, and *Catadiscus*, found in amphibian and reptilian hosts. In 1910 Stiles and Goldberger while proposing a new system of classification created a new superfamily *Paramphistomoidea* to include the forms hitherto classed as amphistomes and divided it into three families: *Paramphistomidae*, *Gastrodiscidae* for *Gastrodiscus* Luck. and *Homalogaster* Poir. and *Gastrothylacidae* for the four genera *Gastrothylax*, *Wellmanius*, *Carmyerius* and *Fischoederius*, the last three being created by them. This classification, however, received criticism by Braun (1911), Odhner (1912), Looss (1912). Stunkard (1917) also remarked, "The classification of Stiles and Goldberger as pointed out by other authors is based on superficial characters and the elevation in rank of the family is in most cases unwarranted. However, the subfamily *Gastrodiscinae* of these authors, appears to be clearly distinguished by the presence of the large ventral pouch and in my opinion should be retained." Fukui, however, denied the importance of the ventral pouch as a family or subfamily character In his opinion the structure of the oral sucker and the topography of the genital organs bring the genus *Gastrothylax* close to *Paramphistoma* in the subfamily *Paramphistominae.* Maplestone (1923) adhering to Amphistoma Rudolhpi (1801) *e. p.*, Nitsch (1819) recognised the super family *Paramphistomoidea* and the three families *Gastrothylacidae, Paramphistomidae*, and *Gastrodiscidae.* This has been criticised by Stunkard and Fukui in 1925 and 1929 respectively. Fukui who has given a complete historical review of the classification, has divided the subfamily *Paramphistominae* into three tribes *Paramphistominea* (without oral evagination), *Stephanopharynginea* (with a single oral evagination) and *Pseudodiscinea* (with paired oral evagination). In my opinion the presence or absence of single or paired oral evaginations as

an important character distinguishing the various genera renders the creation of tribes within the *Paramphistominae* unnecessary.

Stiles and Goldberger (1910) divided *Gastrothylax* Poir. into four genera *Gastrothylax*, *Fischoederius*, *Carmyerius* and *Wellmanius* of which the last two were held synonymous by Maplestone (1923) with which Fukui agrees with this difference that the latter author reduces them to the rank of subgenera. My study of the two species *G. crumenifer* and *G. elongatus* collected from the sheep and goat of Allahabad, India, lead me to think that the genus *Gastrothylax* Poir. should be retained and that the other two genera or *subgenera*, *i.e.*, *Fischoederius* and *Carmyerius* should be merged into it as the excretory and lymphatic systems of *G. crumenifer* and *G. elongatus*, which are included different subgenera, resemble closely. For similar reasons Fukui reduced the genus *Cotylophoron* Stiles and Goldberger (1910) to the rank of a subgenus and included it with three new subgenera *Buxifrons*, *Paramphistoma*, and *Explanatum* in the genus *Paramphistoma* Fischoeder (1901). The creation of the above-mentioned subgenera of *Paramphistoma* on the basis of the position of testes only, I think, unnecessary. I therefore retain the genus *Cotylophoron* as created by Stiles and Goldberger, (1910).

A large number of stomachs of the "Desi" variety of sheep and goat at Allahabad yielded during the years 1931-33 in the months of July to September amphistomatus parasites of the genera *Gastrothylax* and *Cotylophoron*, the list of which as arranged genera-wise is given in Table 2.

TABLE 2

Genus	Host	Rate of infection	Location
1. *Gastrothylax*			
G. elongatus	Sheep	80 per cent.	Stomach
G. crumenifer	Goat	80 per cent	Stomach
2. *Cotylophoron*			
C. ovatum n sp.	Sheep and Goat	100 per cent.	Stomach
C. orientalis n. sp.	Sheep and Goat	100 per cent.	Stomach
C. elongatum n. sp.	Goat	60 per cent.	Stomach

The present paper contains a report on the occurrence of parasites belonging to the genus *Gastrothylax* Poirier 1883 and the description of three new species of the genus *Cotylophoron* Stiles and Goldberger (1910). For the sake of convenience only one species of Cotylophoron is described in detail and only diagnostic characters of the two species are given; whereas the excretory and lymphatic systems of all of them are, however, described at some length.

I am much indebted to Dr H. R. Mehra for his valuable help and for making necessary corrections in the manuscript.

Report on the occurrence of the species of the genus Gastrothylax

Hitherto no study of the genus *Gastrothylax* has been reported from Sheep and Goat of India or any other part of the world Maplestone, however, reported the occurrence of *G. crumenifer* in the Indian bullocks, while several other workers have recorded the occurrence of the various species of this genus from the cattle other than Sheep and Goat in different parts of the world Both Sheep and Goats are found heavily infected at the same time with one species of *Gastrothylax* along with one or two species of *Cotylophoron.* In some goats, however, all the species of *Cotylophoron* are met with along with a species of *Gastrothylax.* The number of these parasites usually varies from hundred to two hundred; in two cases the number varied from fifteen to twentyfive only It is interesting to point out that the species of *Gastrothylax* from goat agreed in all anatomical details with *G. crumenifer* Creplin (1847), Poirier (1883); and that from sheep with *G. elongatus* Poirier. The excretory and lymphatic systems conform almost to the same plan as given by Fukui (1929).

Key to species of Gastrothylax Poirier (1883)

1.	Cephalic half of uterus and vas deferens separated on opposite sides	*G. crumenifer*
	Cephalic half of uterus and vas deferens both median	2
2.	Testes side by side, lobed	3
	Testes obliquely dorsoventral, lobed	4
3.	Genital aperture anterior to the aperture of the ventral pouch	*G. exoporous*
	Genital aperture concealed in the pouch ...	5
4.	Caeca short, ending at about middle of body ...	*G. elongatus*
	Caeca long, reaching acetabulum	*G. cobboldi*

5. Excretory pore and pore of Laurer's canal common, *G. wenyoni*
   Excretory pore and pore of Laurer's canal separate, 6
6. Caeca reaching acetabulum ... ... *G. spatiosus*
   Caeca not reaching acetabulum ... ... *G. gragarius*

Generic diagnosis of Cotylophoron.

Paramphistominae: Oesophagus with or without muscular thickening, caeca long wavy, ending in acetabular zone. Acetabulum of moderate size, terminal, tilts ventrad. Excretory pore pre or post-vesicular. Genital sucker near or behind intestinal bifurcation. Genital papillae present. Testes smooth or lobed, tamdem or oblique. Ovary median or submedian, post-testicular. Laurer's canal may or may not cross the excretory canal. Vitellaria lateral or spreading throughout the body. Excretory system simple with a canal on each side or simple H-shaped or H-shaped with two canals on each side. Lymphatic system simple or branched.

Type species.—*Cotylophoron cotylophorum* Fischoeder (1901).

Cotylophoron ovatum, n. sp.

During the period from July to September an equal percentage of sheep and goat harboured this species in an enormous number. The worms, in living condition, while attached to the stomach villi, are white with a reddish-brown patch in the middle showing no movement of contraction or expansion; but when placed in salt solution they show antero-posterior movements of contraction and expansion keeping alive for more than sixty hours.

The body is dorsoventrally thick and oval in outline, measuring 4·32—6·63 mm. in length and 2·20—2·30 mm. in its maximum breadth which lies in the middle of the body length. The body wall is devoid of spines; but in the anterior half from the cephalad end to the level of the intestinal bifurcation it is thrown into small protuberances or papillae which are broad at the base and bluntly pointed at the end.

The terminal oral sucker, 0·38—0·61 mm. in length and 0·326—0·442 mm. in its maximum breadth, has its lumen narrow dorsoventrally but much broad transversely. The papillae, present internally in the oral sucker of *G. indicum* are absent in this species. The oesophagus with thick muscular walls measures 0·368—0·53 mm. in length and 0·88 -1·16 mm. in its maximum breadth near the intestinal bifurcation. The limbs of the intestinal fork are almost at right angles to the oesophagus. The

intestinal diverticula which are comparatively thick have an almost zigzag course extending up to the cephalad margin of the acetabulum or a little behind it Posteriorly situated acetabulum, 0·526 0·90 mm. in diameter, is completely tilted ventrad.

The main reproductive organs are situated in the posterior two-thirds of the body. The testes of almost equal size are rounded and lie close behind each other. Their zones do not overlap as in *C. indicum.* The anterior testis measures 0·884—1·159 mm in length and 1·05—1·41 mm. in breadth and the posterior testis, 0·84—1·284 mm. in length and 1·00—1·39 mm. in breadth. The vasa efferentia arise from about the middle of the anterior margin of the testes and unite a little anterior to cephalic testis in the median line to form the vas deferens which enters the transversely coiled thin walled vesicula seminalis. The latter passes into the highly muscular and transversely coiled pars musculosa The pars prostatica, 0·90mm. in length, is well developed and is almost obliquely situated in the ventral half of the body. The short ductus ejaculatorious unites with the metraterm to form the ductus hermaphroditicus which traverses the centre of the genital sucker between the genital papillae to open to the exterior. The genital sucker, 0·24—0·32 mm. in diameter, lies 0·16 mm. behind the intestinal bifurcation. The genital pore is of *Paramphistome cervi* type as mentioned by Fukui.

The ovary and shell glands lie in the post-testicular zone between the caudal testis and the dome of the acetabulum. The ovary with an entire outline is situated slightly to the right side, slightly broader than long, measuring 0·22–0·42 mm. in length and 0·33—0·42 mm. in breadth The shell gland mass is almost median and situated just behind the ovary. The uterus, which extends from the ventral pole of the shell gland mass to the genital sucker, runs transversely towards the left intestinal caecum and then passes forwards inside the latter till it reaches in region between the two testes, where it forms two or three transversely arranged coils on the dorsal side, to continue its forward course to enter the metraterm. The eggs measure 0·13—0·132 mm. in length and 0·069 mm. in breadth.

The vitellaria are highly developed, composed of closely set or scattered follicles of 0·30—0·71 mm. length and 0·32 mm. breadth, and almost extracaecal extending from about the intestinal fork to the middle of the acetabulum; sometimes they overlap the caeca almost completely both on dorsal and ventral sides and may even enter the intracaecal area. The vitelline ducts run longitudinally converging centrad at about the middle of the posterior testis, the right one

situated outside the ovary and the shell gland mass, and unite ventrally just behind the latter close in front of the acetabulum.

The Laurer's canal is slightly coiled and opens dorsally in the median line a little anterior to the excretory opening.

The excretory system is simple. The excretory bladder is rounded and situated submedian between the shell gland mass and the anterior margin of the acetabulum, sometimes entering a little behind the latter. The collecting ducts run one on each side, between the intestinal caeca and the body wall almost parallel to the latter from about the caudal margin of caeca to the level of the middle of the oral sucker.

The lymphatic system is also simple conforming almost the same configuration as that of the excretory collecting ducts.

Host—Sheep and Goat.

Habitat—Stomach.

Locality—Allahabad (India).

Cotylophoron orientalis n. sp.

Body elongated, bluntly pointed at cephalic and rounded at posterior ends, 7·56—9·35 mm. in length and 2·98—3·16 mm. in its maximum breadth. Cutical smooth and devoid of protuberances or papillae as present in *C. ovatum* n.sp. Terminal oral sucker 0·68—0·935 mm. in diameter. Oesophagus highly muscular 0·54—0·73 mm. in length with uniform width of 0·17—0·25 mm. Intestinal bifurcation, acute angled situated 0·68—0·85 mm. anterior to the genital sucker of 0·25—0·34. mm. diameter.

Testes irregularly lobed, oblique, separated from each other by a short distance, intracaecal, situated in the third quarter of body length; anterior testis 0·59—1·39 mm. in length and 0·71—1·31 mm. in breadth, posterior testis 0·81—1·36 mm. in length and 1·60 mm. in breadth. Vasa efferentia arise from outer margins of the testes on opposite sides and unite a little anterior to the anterior testis. Coiled vas deferens shorter than pars musculosa; pars prostatica highly developed.

Ovary, median or submedian (0·255—0·48 by 0·34—0·51 mm.) slightly broader than long and situated in first half distance between posterior margin of caudal testis and acetabulum. Shell gland mass is median a little behind ovary. Laurer's canal present. Uterus much coiled behind ovary, less coiled in front running irregularly in median line. Vitellaria of moderate size, *i.e.*, less highly developed than in *C. ovatum* n.sp. extending from about oesophagus to about one-third of acetabulum and coalescing

in the region of genital sucker and post-ovarian zone. Ova 0·136 by 0·085 mm.

Excretory vesicle, median and transversely oval with a short wide transverse canal originating from it on each side which divides into three branches of almost equal caliber, one short running posteriorly and other two long running forwards on each side Outer one of the anterior branches runs cephalad in an annular manner along with intestinal caecum of that side; inner one runs cephalad almost parallel to body wall as far as a little distance in front of genital sucker, where it unites in an arc with that of the other side. More than six branches are given off anteriorly from the arc of which outermosts runs obliquely.

Two lymphatic ducts present on each side of which outer one runs in an annular manner along with intestinal caecum of that side and inner one runs almost parallel to body wall.

Host—Sheep and Goat.

Habitat—Stomach.

Locality—Allahabad (India).

Cotylophoron elongatum n.sp.

Body elongated 11·58—15·30 mm. long and 2·93—4·65 mm. wide in acetabular zone. Cutical smooth and devoid of papillae. Oral sucker terminal, 0·75—0·935 mm. in diameter. Oesophagus muscular, 0·76—1·07 mm. long. Intestinal bifurcation acute-angled about 1·36 mm. anterior to genital sucker of 0·34—0·59 mm. diameter. Intestinal caeca zigzag up to posteriorly situated acetabulum of 1·97—2·38 mm. diameter.

Testes lobed, tandem, broader than long, of almost equal size (1·68—2·04 by 1·78—2·12 mm.) situated in middle third of body length; vas deferens with ciliated epithelium formed between two testes and running cephalad in a zigzag manner; pars musculosa much coiled; pars prostatica highly developed.

Ovary submedian 0·68—0·85 mm. in diameter, situated to the right side almost equidistant from caudal margin of posterior testis and anterior margin of acetabulum. Shell gland mass behind the ovary; Laurer's canal present opening a little anterior to excretory pore. Vitellaria, lateral extending from middle of oesophagus to anterior margin of acetabulum, coalesce behind the shell gland mass and around genital sucker. Uterus narrow and coiled behind ovary but wide in its forward course which is median. Eggs 0·136—0·17 by 0·068—0·085 mm.

Excretory vesicle large, median, situated between ovary and acetabulum, excretory canals one on each side, intracaecal and almost

parallel to body wall, slightly converging centrad a little behind genital sucker and where they are connected by a transverse canal with a short median branch given off in front to genital sucker and extend as far as middle of oral sucker.

Lymphatic system complex highly raticulate throughout the body length.

Host—Goat.

Habitat—Stomach.

Locality—Allahabad (India).

Remarks.—Cotylophoron cotylophorum (Synonym *C. indicum* Stiles and Goldberger, 1910) Fischoeder (1901), as already known differs remarkably from all the species of the genus by the presence of a crossing of the exeretory and Laurer's canals. *Cotylophoron ovatum* n. sp. resembles *C. minutum* Leiper and C. sellsi Leiper, in the shape of the testes; but it differs from them in the size of the body, in the size and arrangement of the vitellaria and in the shape of the excretory bladder. It also differs from *C. orientalis* n. sp. and *C. elongatum* n. sp. in the size and the shape of the body, in the diameter and position of the genital sucker, in the presence of papillae on the body wall in having entire testes, in the size and arrangement of the vitellaria and in the presence of simple excretory and lymphatic systems. *C. orientalis* n. sp. resembles *C. elongatum* n. sp. in the shape of testes and arrangement of the vitellaria but it differs in the size of the body, position of the testes and in the excretory and lymphatic systems.

Key to the species of the Genus Cotylophoron

1.	Excretory canal crossing Laurer's canal ...	*C. cotylophorum.* Fischoeder 1901.
	Excretory canal not crossing Laurer's canal	2
2.	Testes smooth	3
	Testes lobed	4
3.	Testes much smaller than acetabulum ...	*C. minutum.* Lieper, 1910.
	Testes much larger than acetabulum.	
	(i) longer than broad	*C. sellsi.* Leiper, 1910.
	(ii) broader than long	*C. ovatum* n. sp.

4. Testes smaller than acetabulum, situated obliquely ... ... *C. orientalis* n. sp.
Testes larger than acetabulum, situated in tandem ... ... *C. elongatum.* n. sp.

Explanation of Figures

Fig. 1. Dorsal view of *Cotylophoron ovatum* n. sp.
Fig 2. Transverse section of *C. ovatum* n. sp. passing through the region of genital sucker.
Fig. 3. Dorsal view of *Cotylophoron orientalis* n. sp.
Fig. 4. Dorsal view of *Cotylophoron elongatum* n. sp.

Explanation of Lettering

a. acetabulum; g. p. genital papilla; g. s genital sucker; int. intestinal caecum; oes. oesophagus; o. s. oral sucker; ov. ovary; pars m. pars musculosa; pars. p. pars prostatica; s. g. shell glanes mass; t_1. anterior testis; t_2 posterior testis; ut. uterus; vas. ef. vas efferens; vt. vitellaria.

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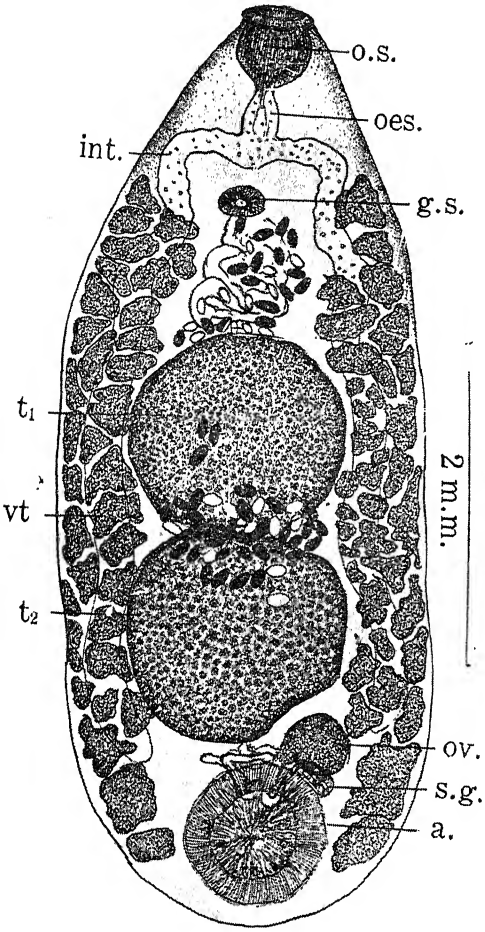

Fig. 1

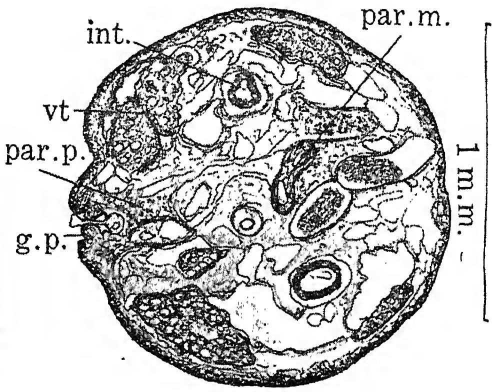

Fig. 2

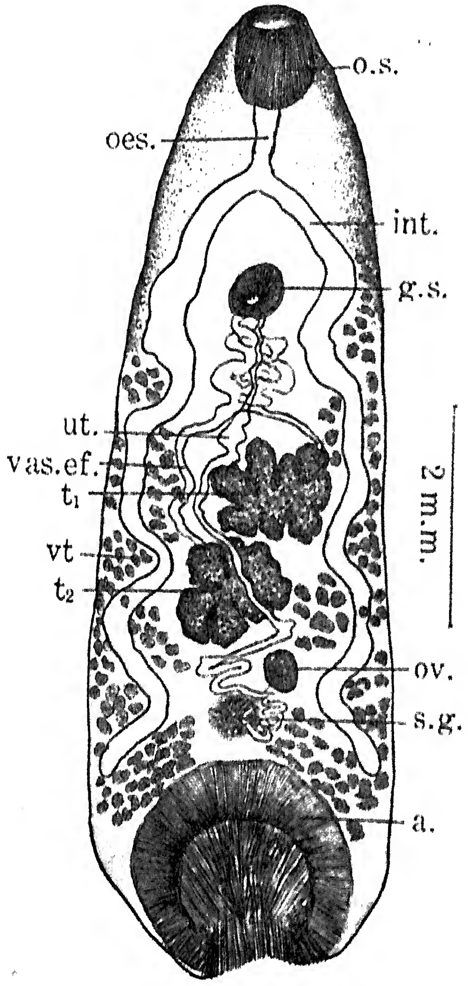
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Fig. 3

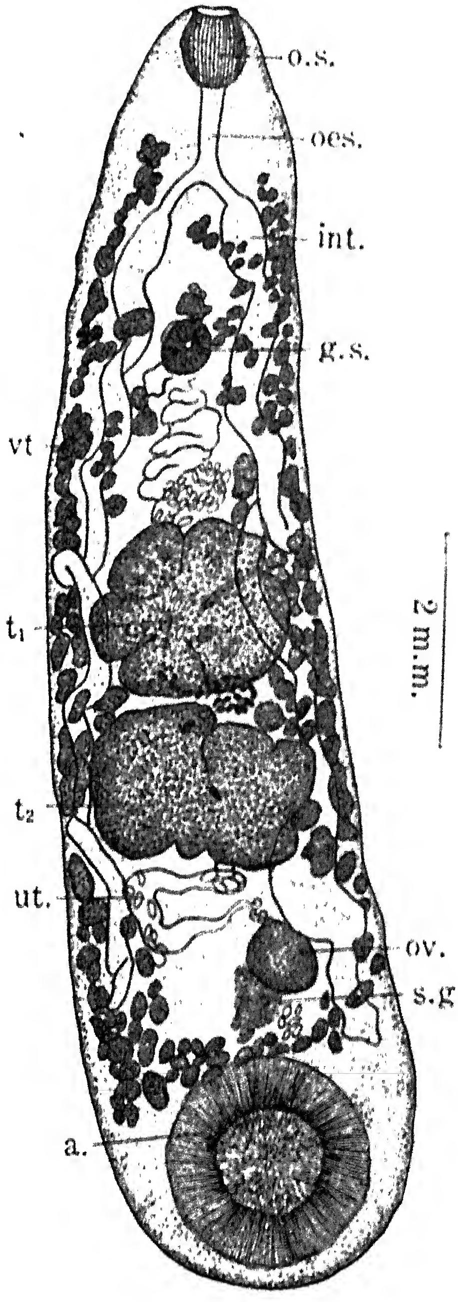
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Fig. 4

ON A NEW TREMATODE FROM AN INDIAN FRESH-WATER FISH

By B. P. Pande

Zoology Department, Allahabad University

Communicated by Dr. H. R. Mehra

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Introduction

The trematode described in this paper was found by me in one out of three specimens of a fresh-water fish, *Rita buchanani*, from the river Gomti at Jaunpur, U.P. The parasites, which furnished a new and an interesting species of *Orientocreadium* Tubangui, were about thirty in number. They were picked out after pouring hot water over the intestinal wall to which they were firmly attached. In living condition the anterior end of the worm was seen to be capable of considerable movement while the posterior part filled with a large number of eggs did not show any movement. One specimen of *Opisthorchis pedicellata* Varma was met with in the gall-bladder of the second fish.

I am deeply indebted to Dr. H. R. Mehra, under whom I am working for his valuable assistance. I thank Dr. D. R. Bhattacharya for allowing me the reagents and other accessories necessary for field work to Jaunpur, a trip to which place was made in early November, 1933.

Orientocreadium indicum, n. sp.

The distomes are moderately small in size, being 1·9-2·6 mm. in length and 0·56-0·66 mm. in maximum breadth which lies in front of equator of the body in the region of the acetabulum. The body is elongated in shape being more tapering at the posterior than at the anterior end. The body-wall is spinose but the distribution of the small backwardly pointed spines is unequal on the body surfaces, those on the ventral side extending little more posteriorly and finally disappearing behind the middle of the posterior testis, while spines on the dorsal side begin to diminish in number near the middle of the ovary being absent altogether posterior to the anterior half of the anterior testis.

The oral sucker, subterminal and circular in outline, measures 0·17—0·21 mm. in diameter with its opening directed towards the ventral surface. The ventral sucker, slightly larger than the oral sucker, measures 0·19—0·23 mm. in diameter and is situated at one-third of the body-length from the anterior end or it may be a little anterior to it. The mouth leads into a well developed prepharynx, followed by a pharynx, 0·1—0·12×0·12—0·14 mm. in size and oval in outline. A very short oesophagus, 0·03 mm. long is present. The intestinal caeca terminate slightly in front of the posterior end of the body. Quite a large number of unicellar oval glands with prominent nuclei are present in the anterior part of the body. They lie chiefly lateral to the pharynx and the intestinal caeca in the preacetabular region and are probably of the nature of unicellular cutaneous glands. These glands are also present intercaecally in front of the ventral sucker and around the metraterm.

The excretory pore is situated terminally at the posterior tip of the body. It leads into a simple elongated bag-shaped excretory bladder which extends up to the middle of the posterior testis. The bladder lies, in the region of testis, ventral to it; while posterior to it its position is ventral to the uterus.

The common genital pore lies immediately in front of the acetabulum and is median in position. There is a small genital atrium.

The testes lie in the posterior half of the body one behind the other in the median line. They are dorsally situated and have entire margins. The anterior testis, situated immediately behind shell-gland and measuring 0·25—0·37 x 0·21—0·3 mm. in size, is nearly globular in shape. The posterior testis, larger than the anterior testis, is longer than broad and measures 0·37—0·44 x 0·21—0·28 mm. in size. The cirrus-sac is large and somewhat half-moon shaped. Its greater part lies slightly to the right of the acetabulum. Its posterior end, extending a little beyond the acetabulum, lies slightly in front of the ovary. The structures contained are a somewhat pear-shaped vasicula seminalis occupying about one-fourth of its length, a well-developed pars prostatica which is slightly bigger than the vesicula seminalis in length, and a long ductus ejaculatorius which is nearly half the length of the cirrus-sac. The space between these structures and the wall of the sac is occupied by prostate gland cells which form a well-developed mass. The long ejaculatory duct is lined internally with forwardly directed spines. A protrusible cirrus is present. Outside the cirrus-sac but lying near its posterior end in the median line between the acetabulum and the ovary and also beside the

latter is the vesricula seminalis externa. It is far greater in size than the vesicula seminalis interna (the ratio in length being 5 : 2). It may be somewhat coiled in appearance and when it is so its proximal part lies ventral to the anterior end of the ovary.

The ovary, globular in shape, measures 0·19—0·23 mm. in diameter and is situated in the posterior part of the anterior half of the body in the median line or a little to the right. Like the testes it is also dorsally placed. The shell-gland complex, nearly equatorial in position, lies to the right side of the ovary between its posterior part and the anterior testis A receptaculum seminis is absent its function being performed by the first few coils of the uterus. The Laurer's canal is present. The uterus, thrown into numerous coils, on emerging from the shell-gland passes to the posterior end of the body. Its coils lie ventral to the testes, the intestinal caeca even extending lateral to the latter with only the vitelline follicles between them and the body wall. On reaching the posterior end the uterus is thrown into similar ascending coils and winds its way forwards lying ventral to the ovary and external seminal vesicle. It ends in a capacious muscular metraterm. The metraterm like the ejaculatory duct is lined internally with relatively large spines and opens externally in the genital atrium. The eggs are numerous, operculated, yellow in colour and measure 0·031 × 0·018 mm. in size.

The vitellaria consist of a large number of follicles beginning at the level of the posterior end of the acetabulum and extending behind the blind end of the intestinal caeca terminate at the posterior end of the body, where the follicles from the two sides meet in the midventral line below the excretory bladder. The follicles are confined mainly to the outer side of the acetabulum, caeca. and uterine coils. In the region of the posterior testis a few follicles may lie dorsal to the caeca and also they extend to midventral line and unite there below the excretory bladder.

The genus *Orientocreadium* was created by Tubangui in 1931 for the trematode commonly obtained from *Clarias batrachus*, of the Philippine Islands, with *C. batrachoides* as the type species. *O. indicum* is the second species of the genus and the first to be described from India. The points wherein the new species differs from the type species and which are considered specific are:—

1. Presence of unicellular cutaneous glands in the preacetabular part of the body in the new species.

2. Maximum breadth in the pre-equatorial part of the body instead of the post-equatorial (as in the case is the type species).

3. Presence of an internal seminal vesicle somewhat pear-shaped in outline (in the type species the large seminal vesicle is located outside cirrus-sac).

4. Ejaculatory duct lined internally with spines unlike the type species.

5. Ovary spherical in shape instead of being oval as in the type species.

6. Metraterm well-developed and also armed with spines which is not the case in the type species.

7. Vitelline follicles are not united to form a lattice work as we find in the type species, vitellaria extending a little more forwards than in the type species.

To accommodate the new species it has become necessary to modify the generic diagnosis of *Orinetocreadium* The emended diagnosis is as follows:—

Allocreadiidae; small distomes with moderately developed suckers; oral sucker subterminal; acetabulum near one-third of body length from anterior end; cuticle spinose; unicellular cutaneous glands may or may not be present in preacetabular part. Pharynx separated from subterminal mouth by short prepharynx; oesophagus very short; intestinal caeca long reaching to near posterior end of body. Genital pore median, immediately preacetabular. Testes median, tendem, postovarian postequatorial; cirrus-sac large, muscular, beside acetabulum; seminal vesicle large entirely outside the cirrus-sac or with a small vesicula seminalis interna; pars prostatica well-developed; ductus ejacualtorius long with or without its internal wall lined with spines. Ovary postacetabular, near the median line and pretesticular. No receptaculum seminis, first few coils functioning as sperm-reservoir. Laurer's canal present. Uterus long, reaching to posterior end in descending coils, then turning forwards may or may not end in a metraterm which is armed like ductus ejaculatorius. Vitellaria moderately developed in follicles extending to posterior end of body and anteriorly do not extend beyond acetabulum. Eggs small and numerous. Parasites of fishes.

Type species—*Orientocreadium batrachiodes* Tubangui, 1931. Key to the species:—

1. Seminal vesicle lying wholly outside the cirrus-sac—
   O. batrachoides Tubangui.
2. Part of seminal vesicle enclosed inside the cirrus-sac—
   O. indicum n. sp.

Systematic Position

While referring his new genus to the family Allocreadiidae Stossich (1904, as emended by Winfield in 1929) Tubangui pointed out that *Orientocreadium* bears a certain resemblance to *Plesiocreadium typicum* Winfield (1929) but differs from it in having its seminal vesicle completely external to the cirrus-sac, in the presence of prepharynx, in having a short oesophagus and in the posterior extent of the vitellaria. He does not say here as to which sub-family his genus fits in. In 1933 he places *Orientocreadium* in the sub-family Allocreadinae Looss 1902.

The strongest point against the inclusion of *Orientocreadium* in Allocreadiinae appears to me to be the nature of the uterus which is certainly more extended and the very large number of eggs than is the case in that sub-family.

Since Winfield's description of the type species of *Plesiocreadium*, a second species has been described by Hunter (1932) from fresh-water fish and designated *P. parvum*. In 1932 Van Cleave and Mueller in course of their description of a new species of *Macroderoides* Pearse (1924), *M. flavus*, have considered *Plesiocreadium* as a synonym of *Macroderoides*. But Hunter, after pointing out the differences between his species and the type species, says that "*P. parvum* superficially resembles the genus *Macroderoides*, and a careful study of sectioned material dispels this delusion and reveals more features characteristic of the genus *Plesiocreadium*."

To me it appears that the genus *Orientocreadium* would belong with *Plesiocreadium* to the sub-family *Plesiocreadiinae* Winfield. The points of difference pointed by Tubangui in his earlier paper such as the character of the seminal vesicle, the extent of the vitellaria and the presence of prepharynx which are sufficiently important should now be considered in the light of the account of the two new species, one of *Plesiocreadium* and the other of *Orientocredium*.

In *Plesiocreadium* Winfield (1929, Char. emend., Hunter 1932) the seminal vesicle is large inside cirrus-sac while in *Orientocreadium batrachoides* it is wholly external and in *O. indicum* it is partly internal and partly external The vitellaria in both the species of *Plesiocreadium* do not extend anteriorly beyond the acetabulum but posteriorly they reach beyond the posterior testis and terminate near the hinder end of the body in *P. parvum* only—a condition which is also met with in the species of *Orientocreadium*. The conditiom of the vitellaria in *P. parvum* comes nearer to that of the genus *Orientocreadium*. A distinct prepharynx is present in *P. parvum* and not in *P. typicum*. A prepharynx is present

in *Orientocreadium*. The fourth point of difference which remains is about the shortness of oesophagus in *Orientocreadium*.

EXPLANATION OF PLATE (Figs. 1—3)

1. Ventral view of an extended specimen.
2. Transverse section through cirrus-sac and external seminal vesicle.
3. Transverse section through ventral sucker.

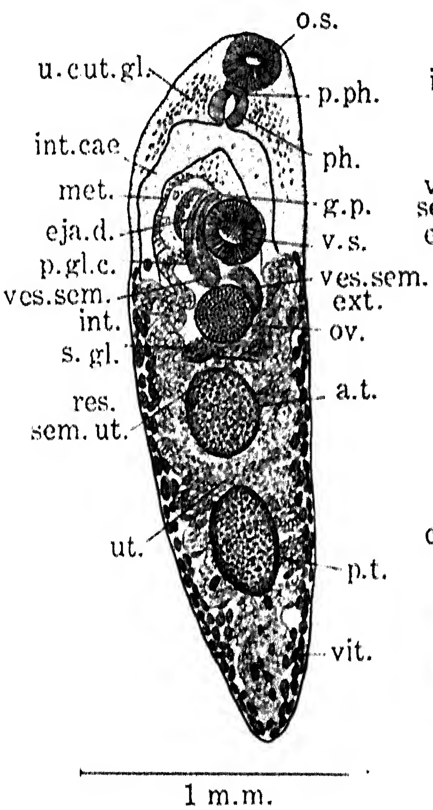

Fig. 1

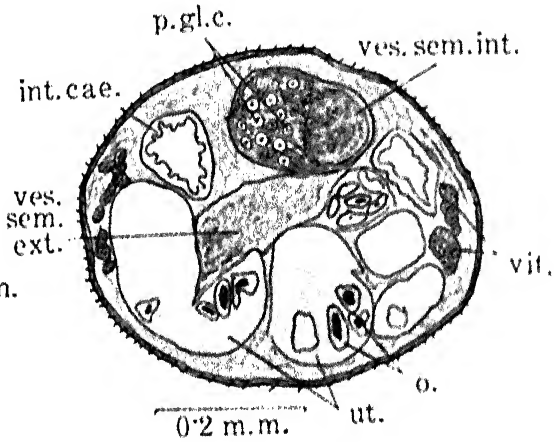

Fig. 2

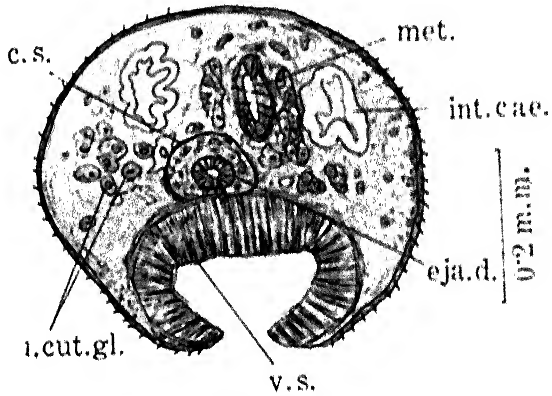

Fig. 3

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ON NEW TREMATODES OF FROGS AND FISHES OF THE UNITED PROVINCES, INDIA

Part IV. The Occurrence and Seasonal Incidence of Infection of Certain Trematodes in the Above Hosts

By Har Dayal Srivastava

Zoology Department, Allahabad University

Communicated by Dr. H. R. Mehra

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An attempt was made during a short holiday trip to Sitapur, Oudh, in the summer vacation of 1932 to make a collection of Helminths of cold-blooded vertebrates. With a view to augment this and to study the phenomenon of seasonal infection of certain parasites under observation, several trips were subsequently made to Sitapur from time to time. Besides, a large number of frogs and fishes available at Allahabad, Lucknow, Nagina and Garhmukteshwar have also been examined for parasites. From this collection which included Trematodes, Cestodes, Nematodes and Acanthocephala, the trematodes of frogs and fishes, of which the greater majority were new to science and have subsequently been described by me in the first three parts, were selected for special study.

For this work more than 500 hosts have been examined. The common-pond frog *Rana cyanophlyctis* of Sitapur has been found to furnish a large number of interesting Trematodes. The two new species of *Halipegus*, *H. mehransis* and *H. spindale* were obtained from this frog, one from the stomach and the other from the intestine. In the stomach was also found *Ganeo gastricus* Srivastava. I also obtained three species of *Pleurogenes*, *P. gastroporus* var. *equalis* Mehra and Negi, *P. sitapuri* Srivastava and *P. orientalis* Srivastava and two species of *Prosotocus*, *P. indicus* Mehra and Negi, and *P. infrequentum* Srivastava from the duodenum of this frog. The intestines were at times found chocked with specimens of *Tremiorchis ranarum* Mehra and Negi, *Ganeo tigrinum* Mehra and Negi and *G. attenuatum* Srivastava, while to the walls of the rectum were found firmly attached specimens of a new species of *Diplodiscus* Dies. Besides an interesting form for which a new genus, *Mehraorchis*, had to be created was obtained from

cysts in the body cavity of the same host. In addition to the forms already reported by Mehra and Negi, *Rana tigrina* of Sitapur was found to harbour a new variety of *Halipegus mehransis* var. *minutum*. This host also showed accidental infection with the species of *Diplodiscus* which is commonly parasitic in *R. cyanophlyctis*. All the trematodes which have been described from *R. tigrina* by Mehra and Negi have also been met with in *R. cyanophlyctis*. But an extensive examination of a large number of hosts from different localities showed a marked host specificity in the case of most of the parasites described by me from *R. cyanophlyctis*.

Examination of a large number of specimens of *Ophiocephalus punctatus* and *O. striatus* caught at Sitapur, Nagina, Garhmukteshwar, Allahabad and Lucknow yielded two new species of *Progonus*, *P. ovocaudatum* and *P. piscicola*, and two new species of a new genus, *Ophiocorchis*, *O. lobatum* and *O. singularis*. These hosts were also heavily infected with cysts of *Clinostomum* and *Euclinostomum* metacercariae found in the body-cavity and *Holostomum* metacercariae found in the pericardial cavity. Specimens of *Silundia gangetica* a common fish in the river Jumna was found harbouring species of *Gasteros-tomum* in the digestive tracts and cysts of *Holostomum* metacercariae in its gonads.

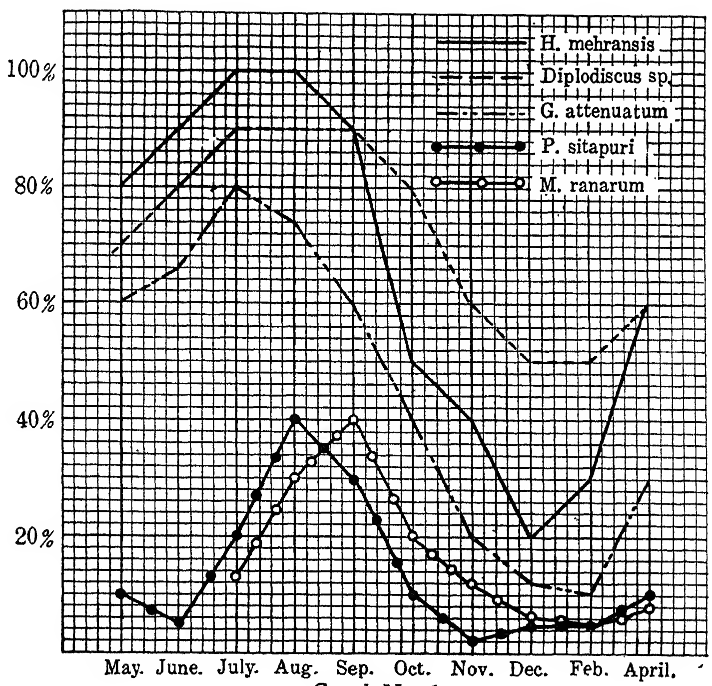
Sewell in 1920 showed " that *Mesocoelium social* Lühe is a common inhabitant of *Bufo melanostictus* during the months of May and June but apparently totally disappears at other times of the year. A similar annual rise and fall occurs in the case of *Fasciola hepatica*, though a few individual flukes may remain in the liver of the affected sheep and it is possible that many trematodes may show the same seasonal variation." The larvae also are seasonal in their appearance as has already been pointed out by Leiper in 1915, Kemp and Gravely in 1919 and Sewell in 1920. I have studied the seasonal incidence of infection of certain trematodes in *R cyanophlyctis* and my observations support the above remark of Sewell. For a study of seasonal incidence of infection it is absolutely necessary that one should confine one's attention to a certain definite locality, for, as has been observed by me, the degree of infection may and often does vary enormously in the same host species obtained from different localities. A remarkable instance of this is found in the fact that whereas specimens of *R. cyanophlyctis* from Sitapur are found heavily infected with a large number of trematodes, those from Allahabad in the same season rarely show any infection at all. During the course of my investigation I have noted the maximum degree of infection as well as the time of the year of such infection. I have found enormous variations in the intensity of infection

with different seasons in the case of five trematodes (*Halipegus mehransis*, *Diplodiscus* sp., *Ganeo attenuatum*, *Mehraorchis ranarum* and *Pleurogenes sitapuri*) even though the hosts were always taken from the same locality. In Graph. No. 1 I have given the results obtained by me from an examination of hosts from Sitapur District at different times of the year. The curves show that the maximum rise in the intensity of infection occurs during the rainy season and the maximum fall in winter months. Graph No. 2 shows the maximum frequency of occurrence in the case of the new trematodes that I have studied.

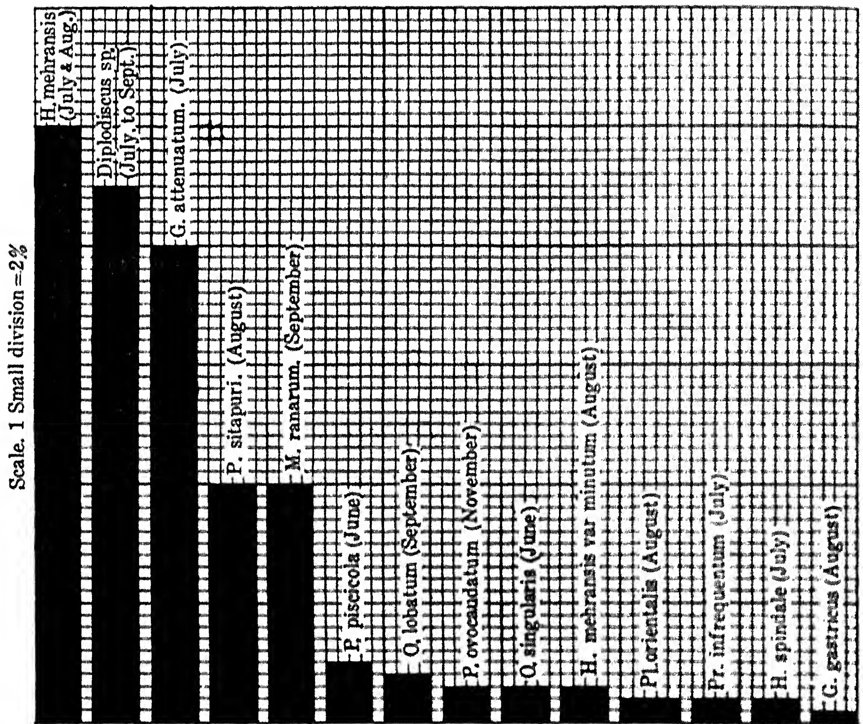
I am greatly indebted to Dr. H. R. Mehra, for his valuable help and advice and to Dr. D. R. Bhattacharya for providing me laboratory facilities during holidays and giving me reagents and other requisites for field collection. I take this opportunity of expressing my deep indebtedness to my brother, Mr. Girja Dayal Srivastava, who has very kindly provided me with a regular supply of the hosts from Sitapur, Lucknow, Nagina and Garhmukteshwar for parasitological examination.

EXPLANATION OF GRAPHS

Graph :—No. 1 : Showing Seasonal Incidence of Infection.
Graph :—No. 2 : Showing the maximum Frequency of Infection.



Graph No. 1



Graph No. 2

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ADDENDUM

Since the communication of the above paper for publication in the Proceedings, Academy of Sciences, U.P., Allahabad, Tubangui's paper on "Trematode Parasites of Philippine Vertebrates, VI" has been received in the Department. In this paper the author gives an account of a new species of *Diplodiscus—D. amphichrus*. In the general topography of its organs and the position of the genital pore Tubangui's species resembles very closely the Indian species referred to in my above paper. The Indian representative, however, presents many important and constant points of difference from *D. amphichrus*, such as, the larger size of its body and the various organs (Table 1), presence of well developed glands round the oesophagus, the character of the oesophageal bulb which is conspicuous and well developed and the disposition of the vitellaria which do not meet mesially in front of the genital pore. These differences are sufficiently important to warrant the creation of a new variety for the Indian species—*D. amphichrus*—var. *magnum*. n. var.

Table I

	Length	Maximum breadth	Length of oesophagus	Size of Oral sucker	Size of Oral pouches	Size of oeso. bulb.	Diameter of Acetabulum
D. amphichrus Tubangui ...	1·6—3·3*	0·6—0·86	0·27—0·44	0·11 - 0·26	0·11 - 0·26	Inconspicuous	0·5 —1·08
D. amphichrus var. *magnus.* N. Var. ...	2·6—6·7	1·2—2·1	0·26 0 56	0·35—0·56	0·1—0·2	0·1—0·26 × 0·1 0·17	0 87 - 1·5

	Accessory sucker	Testis	Cirrus sac	Position of genital pore	Ovary	Eggs.
D. amphichrus Tubangui ..	0·1—0·13	0·14—0·46	0·12—0·22	Close behind the anterior third of body length.	0·12 0·24	104—112 × 62·4 - 70·7 microns
D. amphichrus var. *magnus.* N. Var. ...	0·17—0·35	0·37—0·7	0·1 — 0·15	Do.	0·17—0·4	0·12 0·14 × 0·05—0·7 m. m.

Fig. 1. Ventral view of *Diplodiscus amphichrus* var. *magnus.*

LETTERING

O. s.—Oral sucker. O. p.—Oral pouch. Oes.—Oesophagus. Oes. gl.—Oesophageal glands. Oes. b.—Oesophageal bulb. C. s.—Cirrus sac. G. p.—Genital pore. Eg.—Eggs.

Int. c.—Intestinal caecum. T.—Testis. Vit.—Vitellaria. Ov.—Ovary. S. g.—Shell gland. Acet.—Acetabulum. Acc. s.—Accessory sucker, i.e, a sucker-like structure in the centre of the acetabulum.

LITERATURE

Tubangui, M.A—Philippine Journ. Sci., Manila. Vol. 25, No. 2, Oct. 1933.

* All measurements are in mm.

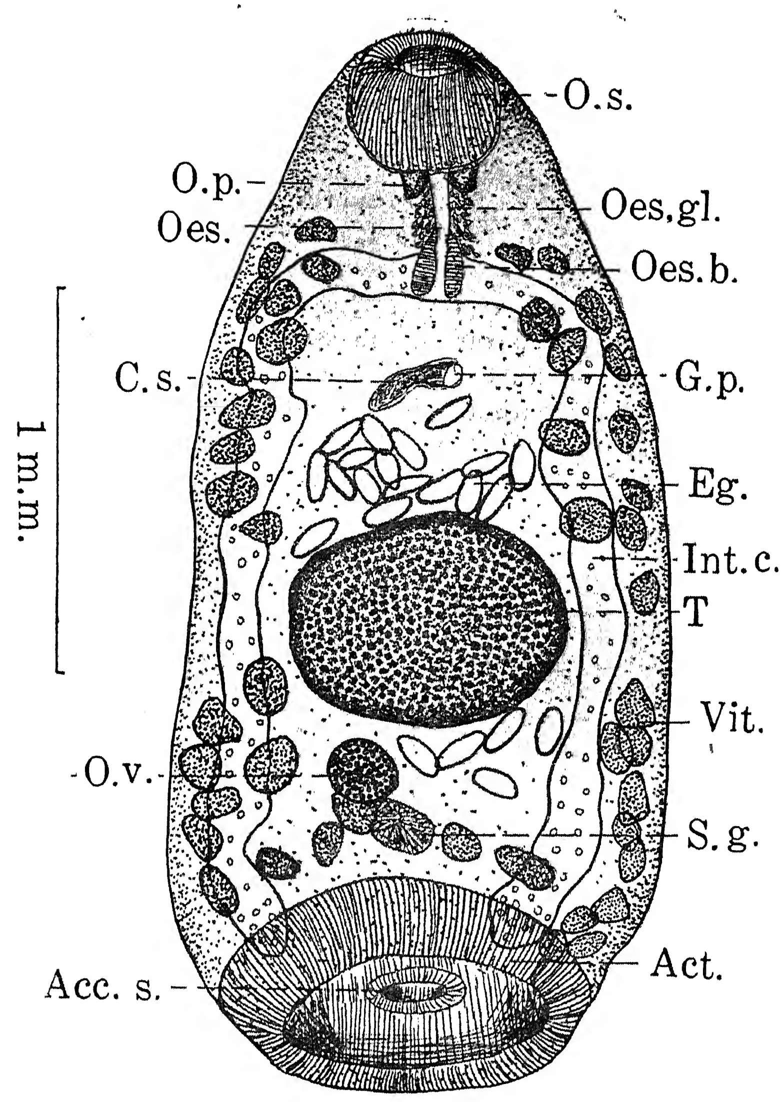

Fig. 1

Errata to Part I Published in the Bull. Acad. Sci. U. P., Vol. 3, No. 1, pp. 41–60, 1933.

Page 41, line 5 from bottom, for sids read sides.

Page 47, line 7, for vntral read ventral.

Page 48, line 13 from bottom, delete "with a polar filament at the posterior end."

Page 48, line 11 from bottom, after "uterine coils" add, "the presence of polar filament (in Halipegus)."

Page 57, line 5, for (Figs. 7 & 8) read (Fig. 7.)

Page 58, lines 3, 4, 5, 6 & 9 (Figs. 1, 4, 5, 6, 7), from bottom, for ventral read dorsal.

Note on species of Halipegus Looss, 1899.

The copulatory apparatus of the species of *Halipegus* described by me in Part I is very much like that of most Hemiurids The voluminous vesicula seminalis is continued into a tubular pars prostatica which is surrounded by numerous prostate gland cells lying free in the parenchyma. Terminally the pars prostatica opens through a fairly long ductus ejaculatorius into the highly muscular metraterm The well developed ductus hermaphroditicus is protrusible and opens on a conical papilla in the genital atrium.

In a joint paper, "Fauna Helminthologica Dos Peixes De Agua Doce Do Brasil." Separado Do Volume 1, Dez. 1928, Dos Archiv. Do Instituto Biologico, Sao Paulo, Travassos, Artigas and Pereira have described three species of *Genarchella* N. Gen. from Brasilian fishes. The genus does not differ in any character, which might be considered of generic importance, from *Halipegus* Looss and should, therefore, be considered synonymous with the latter.